REMARKS ON THE CLASSIFICATION OF
RIEMANN SURFACES

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1. Introduction. In [5] Royden constructed two Riemann surfaces to show that neither of the classes $O_{AB}$ and $O_{AD}$ of Riemann surfaces is quasiconformally invariant. In the present note it is shown how a slight modification of Royden's argument serves to establish the more general

Theorem 1. No class of Riemann surfaces that contains $O_{L}$ and is contained in $O_{AD}$ is quasiconformally invariant.

This result is then used to prove

Theorem 2. There is no inclusion relation between either of $O_{L}$ and $O_{L'}$, and any one of $O_{HD}$, $O_{FD}$ and $O_{FB}$.

2. Notation and background. For any class $T$ of functions that can be considered on a Riemann surface, let $O_{T}$ stand for the class of all Riemann surfaces that do not admit nonconstant members of $T$. Denote by $L$ the class of Lindelöfian meromorphic functions (see [2]), i.e. meromorphic functions of bounded characteristic, and by $L'$ the class of those members of $L$ which are pole-free. $AB$ and $AD$ denoting, respectively, the classes of bounded and Dirichlet-bounded analytic functions, it is known (see [2, p. 442] and [1, p. 201 and p. 256]) that

(1) $O_{L} \subset O_{L'} \subset O_{AB} \subset O_{AD}$.

Let, now, $HD$ denote the class of Dirichlet-bounded harmonic functions and $FB$ and $FD$ signify, respectively, the classes of bounded and Dirichlet-bounded harmonic functions whose conjugate periods vanish along dividing cycles. It is clear that

(2) $O_{HD} \subset O_{FD} \subset O_{AD}$,

and it is known [7, p. 469] that

(3) $O_{FB} \subset O_{FD}$.

3. The examples. Denote by $S$ the region of the complex plane obtained by punching out $z = 0$ and $z = 2$ from the disc $|z| < 3$. Let

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Let \( W_1 \) be a two-sheeted conformal cover of \( S \) having branch points over \( z = 1/n \) and \( z = 2 - 1/n \), \( n = 2, 3, \ldots, \infty \), and such that two curves of \( W_1 \) lie over the unit circle in \( S \). Evidently, the projection map from \( W_1 \) to \( S \) is an AD function, i.e.

\[
W_1 \subseteq O_{AD}.
\]

Choose, now, \( s \), an irrational multiple of \( 2\pi \). Cut the upper sheet of \( W_1 \) over the unit circle \( |z| = 1 \) and identify the point over \( e^{i\theta} \) on the inside of the cut with the point over \( e^{i\theta+i\pi} \) on the outside. Denote the resulting Riemann surface by \( W_2 \). We claim that

\[
W_2 \subseteq O_L.
\]

For, otherwise, let \( f \in L \) on \( W_2 \). The projection map from the subregion \( W'_2 \) of \( W_2 \) over the annulus \( 0 < |z| < 1 \) into this annulus is of constant valence two. Hence, by a theorem of Heins [2, p. 444], \( f \) is the quotient of two bounded analytic functions, say, \( g \) and \( h \), on \( W'_2 \). But then, the argument of Royden [5, p. 6] shows that \( g \) and \( h \) are single valued functions of \( z \) for \( |z| < 1 \), and hence, so is \( f \). Similarly, \( f \) is a single valued function of \( z \) for \( 1 < |z| < 3 \). Hence, over \( |z| = 1 \), we have \( f(e^{i\theta}) = f(e^{i\theta+i\pi}) \).

Iterating we obtain that

\[
f(e^{i\theta}) = f(e^{i\theta+im\pi}), \quad m = 1, 2, \ldots, \infty.
\]

Since the set of points \( \{e^{i\theta+im\pi}\} \) is dense in the unit circle, this implies that \( f \) is a constant, which is a contradiction that establishes (5).

Now, the map \( \phi: W_1 \rightarrow W_2 \), obtained by taking \( \phi \) to be the identity except on the annulus over \( 1/2 < |z| < 1 \) in the upper sheet of \( W_1 \) and \( \phi(re^{i\theta}) = re^{i\theta+i\pi(2\pi-1)} \) on this annulus, is a quasiconformal homeomorphism of \( W_1 \) onto \( W_2 \). This observation, together with (4) and (5), establishes Theorem 1.

**Remarks.** (i) As noted by Royden [5, p. 6], the map \( \phi \) is "ultimately" conformal. Hence, the property of belonging to any of the classes of Theorem 1 is not a property of the ideal boundary [6, p. 58].

(ii) Examples of classes of Riemann surfaces for which Theorem 1 is applicable are \( O_L, O_L', O_{AB}, O_{AD} \) and the classes \( O_{AM} \) [3, p. 179].

(iii) A construction, similar to the one above, carried out over the Riemann sphere instead of the disc \( |z| < 3 \), yields an example to show that the class of parabolic Riemann surfaces admitting meromorphic functions of bounded valence is not preserved under quasiconformal maps.

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\( ^2 \) The author is indebted to the referee for pointing this out.
4. **Proof of Theorem 2.** It is known [6, p. 57] that $O_{FD}$ is quasi-conformally invariant. This, in view of (2) and Theorem 1, implies that

$$O_L \subseteq O_{FD}, \quad O_L \subseteq O_{HD}. \quad (6)$$

Also, there exists [2, p. 441] a planar surface which does not belong to $O_{L'}$ but belongs to $O_{AB}$. Since every cycle on a planar surface is dividing, this implies that

$$O_{FB} \subseteq O_{L'} \quad (7)$$

Finally, in view of Theorem 26H of [1, p. 264] and (1), we have

$$O_{HD} \subseteq O_{L'} \quad (8)$$

Combining the first part of (1) with (6)–(8), we obtain the result.

**Note.** An alternative proof of the second part of (6) was given in [4].

**References**


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