ON FOGUEL’S ANSWER TO NAGY’S QUESTION

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Nagy’s question is whether or not every power-bounded operator is similar to a contraction [3]. (“Power-bounded” means that the norms of the positive powers are bounded.) Foguel’s answer is no [1]. The purpose of this note is to look at Foguel’s ingenious counterexample from a point of view somewhat different from his own. The advantage of the new look is that it is less computational; its drawback is that the intuitive motivation is less transparent.

Let $H_0$ be a Hilbert space with an orthonormal basis \{e_0, e_1, e_2, \ldots \}, and let $S$ be the unilateral shift on $H_0$ ($Se_n = e_{n+1}$, $n = 0, 1, 2, \ldots$). Let $J$ be an infinite set of natural numbers that is “sparse” in the sense that if $i$ and $j$ belong to $J$ and $i < j$, then $2i < j$. (Example: $J$ can be the set of positive integral powers of 3.) Let $Q$ be the projection from $H_0$ onto the span of all the $e_i$’s with $j$ in $J$. If $H$ is the direct sum of two copies of $H_0$ (the set of all ordered pairs $(f, g)$ with $f$ and $g$ in $H_0$), then every operator on $H$ is given by a two-by-two matrix whose entries are operators on $H_0$. Principal assertion: if

$$A = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix},$$

then $A$ is power-bounded, but $A$ is not similar to a contraction.

A trivial induction shows that

$$A^n = \begin{pmatrix} S^{*n} & Q_n \\ 0 & S^n \end{pmatrix},$$

where $Q_0 = 0$ and $Q_{n+1} = \sum_{i=0}^{n} S^{*n-i}QS^i$, $n = 0, 1, 2, \ldots$. To prove that $A$ is power-bounded is the same as to prove that the norms of the $Q$’s are bounded. It turns out, in fact, that each $Q$ is a partial isometry whose range is spanned by a set of $e$’s. To prove this, consider $Q_{n+i}e_m = \sum_{i=0}^{n} S^{*n-i}Qe_{m+i}$. If $n-i > m+i$, then $S^{*n-i}Qe_{m+i} = 0$, because either $m+i \notin J$ (in which case $Qe_{m+i} = 0$), or $m+i \in J$ (in which case $S^{*n-i}$ annihilates $e_{m+i}$). Among the remaining values of $i$ (the ones for which $i \leq n \leq m+2i$) at most one can be such that $m+i \in J$. Reason: if both $i$ and $j$ have these properties, and, say, $i < j$, then $m+i < m+j$, so that $2(m+i) < m+j$, or $m+2i < j$, which

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contradicts the relation \( j \leq n \leq m + 2i \). Conclusion: \( Q_{n+i}e_m \) is either 0 or \( e_{m+2i-n} \); it is the latter just in case there exists an \( i \) (necessarily unique) such that \( i \leq n \leq m + 2i \) and \( m + i \in J \). This conclusion will be used again presently; its function so far was to prove that \( A \) is power-bounded.

It remains to prove that \( A \) is not similar to a contraction. For this purpose Foguel introduces the set \( Z(A) \) of all those vectors \( f \) in \( H \) for which \( A^n f \to 0 \) weakly as \( n \to \infty \). (Here \( H \) can be an arbitrary Hilbert space and \( A \) an arbitrary operator on it.) The pertinent lemma is that if \( A \) is similar to a contraction, then \( Z(A) \cap (Z(A^*))^\perp = \{0\} \). (A proof of the lemma appears below.) The conclusion of the preceding paragraph makes it possible to apply the lemma, as follows. If \( j \in J \), then \( Q_{2j+i}e_0 = e_0 \). Since \( A^{2j+1}(0, e_0) = (Q_{2j+i}e_0, S^{2j+1+i}e_0) = (e_0, e_{2j+1}) \), so that \( A^{2j+1}(0, e_0) \to (e_0, 0) \) weakly as \( j \to \infty \) (through values in \( J \)), it follows that if \( (f, g) \in Z(A^*) \) (that is, if \( A^*(f, g) \to (0, 0) \) weakly as \( n \to \infty \)), then

\[
(\langle e_0, 0 \rangle, (f, g)) = \lim_{j \to J} (A^{2j+1}(0, e_0), (f, g)) = \lim_{j \to J} (\langle 0, e_0 \rangle, A^{2j+1}(f, g)) = 0,
\]

so that \( (e_0, 0) \in (Z(A^*))^\perp \). Since, however, \( A \langle e_0, 0 \rangle = (0, 0) \), the vector \( (e_0, 0) \) belongs to \( Z(A) \) also, and consequently \( A \) cannot be similar to a contraction.

For the lemma Foguel refers to an earlier paper. Here is an alternative approach, via the theory of strong unitary dilations [2].

(1) If \( U \) is unitary, then \( Z(U) \subseteq Z(U^*) \). Indeed, represent \( U \) as multiplication by a measurable function \( \phi \) of constant modulus 1 on some \( L^2(\mu) \). It is to be proved that if \( \int f \phi d\mu = 0 \) for every \( g \), then \( \int \phi f d\mu = 0 \) for every \( h \). To prove it, given \( h \), put \( g = (\text{sgn} f)^h \), and form the complex conjugate of the hypothesis.

(2) If \( C \) is a contraction, then \( Z(C) \subseteq Z(C^*) \). To prove this, let \( U \) be a minimal strong unitary dilation of \( C \). That is: if \( C \) operates on \( H \), then \( U \) operates on a larger Hilbert space \( K \); if \( P \) is the projection from \( K \) onto \( H \), then \( C^nf = Pu^nf \) for all \( f \) in \( H \) \((n = 1, 2, 3, \ldots)\). For each \( f \) in \( Z(C) \), let \( K_f \) be the set of all those \( g \) in \( K \) for which \( (U^nf, g) \to 0 \). Since \( f \in Z(C) \), it follows that \( H \subseteq K_f \); indeed, if \( g \in H \), then \( (U^nf, g) = (C^nf, g) \). It is trivial that \( K_f \) is a linear manifold; the power-boundedness of \( U \) implies that \( K_f \) is closed. Since \( K_f \) is invariant under both \( U \) and \( U^* \), the minimality of \( U \) implies that \( K_f = K \) for each \( f \) in \( Z(C) \). This implies that \( Z(C) \subseteq Z(U) \), and hence, by (1), that \( Z(C) \subseteq Z(U^*) \). Since \( U^* \) is a strong dilation of \( C^* \), it follows that \( Z(C) \subseteq Z(C^*) \).

The promised lemma is now within reach. If \( A \) is similar to a con-
traction $C$, say $A = TCT^{-1}$, then it is easy to verify that $Z(A) = TZ(C)$ and $(Z(A^*))^\perp = T(Z(C^*))^\perp$. Since, by (2), $Z(C) \cap (Z(C^*))^\perp = \{0\}$, the conclusion $Z(A) \cap (Z(A^*))^\perp = \{0\}$ follows by an application of $T$.

REFERENCES


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