ON CLANS OF NON-NEGATIVE MATRICES

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A clan is a compact connected topological semigroup with identity. Professor A. D. Wallace has raised the following question [9]: Is a clan of real \( n \times n \) matrices with non-negative entries, which contains the identity matrix, necessarily acyclic? That is to say, do all of the Alexander-Čech cohomology groups with arbitrary coefficients (in positive dimensions) vanish? In this paper the slightly stronger result, that any non-negative matrix clan is contractible, is obtained. This follows from the result, interesting in itself, that a compact group of non-negative matrices is finite (Theorem 2).

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The set of order \( n \) non-negative matrices is denoted by \( N_n \). The real and complex general linear groups of order \( n \) are represented by \( \text{Gl}(n, R) \) and \( \text{Gl}(n, C) \), respectively. The semigroup terminology used is that of [8]; in particular, \( K \) denotes the minimal ideal of a clan \( S \), \( E \) denotes the set of idempotents of \( S \), and for \( e \in E \), \( H(e) \) is the maximal subgroup of \( S \) containing \( e \). An isomorphism is an isomorphism which is also a homeomorphism. The topology of \( N_n \) is any locally convex topology; for example, the topology of Euclidean \( n^2 \)-space.

The equation \( M = \text{diag}(A, B) \) means that \( M \) is the matrix which, in 2\( \times \)2 block form, has the square submatrix \( A \) in the upper left corner, the square submatrix \( B \) in the lower right corner, and zero entries elsewhere. The \( k \times k \) identity matrix is denoted by \( I_k \) when used as a submatrix. The set of eigenvalues of a matrix \( M \) is denoted by \( \lambda(M) \).

The well-known theorem [1, p. 80] that a non-negative matrix \( M \) has a real eigenvalue \( r \) such that if \( \lambda \in \lambda(M) \), then \( |\lambda| \leq r \) is used without proof. Also used without proof is the following theorem, due to Karpelevich [3], and stated in less than full generality:

**Theorem 1.** Let \( M \in N_n \), and let \( M \) have maximal real eigenvalue 1. If \( \lambda \in \lambda(M) \), \( |\lambda| = 1 \), then \( \lambda^k = 1 \) for some \( k \leq n \).

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Lemma 1. Let $X \in G$, a compact subgroup of $\text{Gl}(n, C)$. If $\lambda \in S(X)$, then $|\lambda| = 1$.

Proof. The determinant function maps $G$ homomorphically into the unit circle. Hence $1 = |\det X| = |\lambda_1 \lambda_2 \cdots \lambda_n|$, $\lambda_i \in S(X)$. Let $P \in \text{Gl}(n, C)$ such that $A = PXP^{-1}$ is triangular, diagonal $A = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Since diagonal $A^t = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ and the group $PGP^{-1}$ is compact, it follows that $|\lambda_i| \leq 1, i = 1, \cdots, n$. This is clearly sufficient.

Theorem 2. Let $H(e)$ be a compact topological group, $H(e) \subset N_n$. Then $H(e)$ is finite.

Proof. Define $f : H(e) \to \text{Gl}(n, R)$ by $f(x) = x + I - e$. The function $f$ is clearly an isomorphism. Since $f(H(e))$ is a compact subgroup of $\text{Gl}(n, R)$, $H(e)$ is a Lie group. The identity component $C$ of $H(e)$ is therefore open; hence it suffices to prove that $H(e)$ is totally disconnected. If $C \neq \{e\}$, then $C$ has a nontrivial one parameter group [5, p. 105], hence elements of infinite order. The proof is then completed by contradiction when it is shown that every element of $H(e)$ has finite order.

Let $X \in H(e)$. There exists $B \in \text{Gl}(n, R)$ such that $B(e)B^{-1} = \text{diag}(I_k, 0)$, where rank $e$ is assumed equal to $k$. Since $B(e)B^{-1}$ is an identity for $BXB^{-1}$, $BXB^{-1} = \text{diag}(X_k, 0)$, where $X_k$ is a rank $k$ real $k \times k$ matrix. Let $f$ be the isomorphism of $BH(e)B^{-1}$ into $\text{Gl}(n, R)$ defined by $f(BXB^{-1}) = BXB^{-1} + I - BeB^{-1}$. Since $f(BH(e)B^{-1})$ is isomorphic to $H(e)$, it suffices to find an integer $m$ such that $f(BXB^{-1})^m = f(BeB^{-1}) = I$.

Assume $k < n$. Note $S(X) = S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$. For if $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$, then det($X_k - \lambda I_k$) = 0. Hence 
\[ \det(f(BXB^{-1}) - \lambda I) = (1 - \lambda)^{n-k} \cdot \det(X_k - \lambda I_k) = 0 \]
and
\[ \lambda \in S(f(BXB^{-1})). \]

Conversely, if $\lambda \neq 1$ and $\lambda \in S(f(BXB^{-1}))$ then $\lambda \in S(BXB^{-1})$. Finally, by Lemma 1, $\lambda \in S(f(BXB^{-1}))$ gives $|\lambda| = 1$; therefore $\lambda \in S(BXB^{-1})$, $\lambda \neq 0$ also yields $|\lambda| = 1$. Since $X \in N_n$, $1 \in S(BXB^{-1})$, and $S(BXB^{-1}) = S(f(BXB^{-1})) \cup \{0\}$. By Theorem 1, $S(BXB^{-1}) \subset \{\lambda : \lambda^t = 1, t \leq n\} \cup \{0\}$. If $k = n$, a similar argument can be given. In either event $S(f(BXB^{-1})) \subset \{\lambda : \lambda^t = 1, t \leq n\}$. Let $P \in \text{Gl}(n, C)$ such that $D = Pf(BXB^{-1})P^{-1}$ is lower triangular and diagonal $D = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Note $\lambda_i \in S(f(BXB^{-1}))$, $i = 1, \cdots, n$. Let $m =$ least common multiple $\{t_i : \lambda_i^t = 1, t_i \leq n\}$. Then diagonal $D^m = \{1, 1, \cdots, 1\}$. Now
if \( j = i - 1 \), then \((D^m)_{ij} = \rho \cdot (D^m)_{ij}\). Hence, by the compactness of
\( Pf(BH(e)B^{-1})P^{-1}, (D^m)_{ij} = 0, j = i - 1 \). By a straightforward induction,
it follows that \((D^m)_{ij} = 0, j < i, i = 1, \ldots, n \). Hence \(D^m = I\), and therefore
\( f(BXB^{-1})\) has order \( \leq m \), which completes the proof.

**COROLLARY 1.** Let \( S \) be a continuum semigroup in \( N_n \). Then \( K \subseteq E \).

**Proof.** Fix \( e \in E \cap K \). Then \( eSe = H(e) \) [8]. Since \( eSe \) is a continuum,
it is degenerate; hence \( H(e) = \{ e \} \). The corollary now follows from the fact that
\( K = \bigcup \{ H(e) : e \in K \} \).

If \( S \) is a clan, it is known [8] that \( H^*(S) = H^*(eSe) \) for \( e \in K \cap E \),
\( n \geq 0 \). If, also, \( S \subseteq N_n \), then by Theorem 2, \( H^*(S) = H^*(\{ e \}) = 0, n > 0 \).
Hence \( S \) is acyclic. It will now be shown that \( S \) is contractible. The
following lemma is due to Gluskin [2].

**LEMMA 2.** Let \( S \) be an \( n \times n \) complex matrix semigroup. Let \( e, f \in E \) and
\( f \in eSe \). If \( f \neq e \), then \( \text{rank } f < \text{rank } e \).

**Proof.** Suppose \( \text{rank } e = r, e \neq f \). Choose \( v \) such that \( vev^{-1} = \text{diag}(I_r, 0) \).
Then \( vfv^{-1} = \text{diag}(g, 0) \), since \( e \) is an identity for \( f \). Note
\( g \) is an \( r \times r \) complex matrix, and \( g^2 = g \). Since \( \text{rank } vfv^{-1} = \text{rank } f \), it
suffices to show \( \det(g) = 0 \). If this is not the case, then \( g \) is an idempotent
in \( \text{Gl}(r, \mathbb{C}) \); hence \( g = I_r \). But this implies \( f = e \), contrary to
assumption. This completes the proof.

An \( I \)-semigroup is a clan on an interval such that one endpoint is
an identity and the other a zero. It is shown in [6] that the only types
of \( I \)-semigroups are the following: (i) \( S \) has the multiplication of the
real interval \([0, 1]\); (ii) \( S \) has a multiplication isomorphic to the interval
\([1/2, 1]\) under the operation \( x \circ y = \max \{1/2, xy\} \); (iii) \( S \) is
idempotent and has a multiplication isomorphic to the interval \([0, 1]\)
under the operation \( x \circ y = \min \{x, y\} \); (iv) \( S \) is the union of a collection of semigroups of types (i), (ii), and (iii) which meet only at
their respective endpoints.

**LEMMA 3.** Let \( S \) be a clan in which, for each \( e \in E \), \( H(e) \) is totally dis-
connected. Suppose also that there exists a neighborhood \( V \) of 1 such that
\( V \cap E = 1 \). Then there is an \( I \)-semigroup in \( S \) having 1 as an identity.

**Proof.** It is well known [7] that the existence of the neighborhood
\( V \) above is sufficient to insure a local one-parameter semigroup
\( \sigma([0, 1]) \) in \( V \) such that \( \sigma(0) = 1, \sigma(a) \in H(1), 0 < a \leq 1 \), and if \( \sigma(a) = \sigma(b)g, g \in N(1) \), then \( a = b \) and \( g = 1 \). In the same paper, it is shown that
\( \sigma \) can be extended to a full one-parameter semigroup by defining
\( \sigma(t) = \sigma(1) \sigma(t - 1) \) for \( t \in [1, 2] \) and proceeding inductively. Now the
closure of \( \sigma([0, \infty)) \) is a commutative clan, hence its minimal ideal is
a connected group, and therefore a single point. It follows by a theorem of Koch [4] that this clan has exactly 2 idempotents and is an \( I \)-semigroup.

**Theorem 3.** Let \( S \) be a nondegenerate clan in \( N_n \). Then \( S \) contains an \( I \)-semigroup from 1 to \( K \), and \( S \) is contractible.

**Proof.** By Lemma 2, there exists a neighborhood \( V \) of 1 containing no other idempotents; this follows from the fact that the rank of an idempotent equals its trace. By Theorem 2 each \( H(e) \) is finite. It follows from Lemma 3 that there exists an \( I \)-semigroup from 1 to \( e \in E \). By Lemma 2, rank \( e < \text{rank 1} \). If \( e \in K \), then \( eSe \) is a nondegenerate subclan with identity \( e \), and the above argument produces an \( I \)-semigroup from \( e \) to \( f \in E \), rank \( f < \text{rank e} \). In this manner, an idempotent of minimal rank in \( S \) is obtained, which clearly belongs to \( K \). The union of the \( I \)-semigroups constructed above is the desired \( I \)-semigroup.

Let \( T \) be an \( I \)-semigroup in \( S \) with endpoints 1 and \( e \in K \cap E \). Define \( F: S \times T \rightarrow S \) by \( F(x, t) = txt \). Then \( F(x, 1) = x \), and \( F(x, e) = exe = e \), for each \( x \in S \). Hence \( S \) is contractible. This completes the proof.

By Lemma 2, no \( I \)-semigroup in \( N_n \) can be of type (iii) mentioned above. On the other hand, it is well known that if \( A \) is a nilpotent \( n \times n \) complex matrix, then \( A^n = 0 \). It follows that the \( I \)-semigroups in \( N_n \) are either of type (i), or of type (iv), constructed by joining together the endpoints of semigroups of type (i).

**Bibliography**


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