LINEAR SYSTEMS OF REAL QUADRATIC FORMS

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1. It is natural to consider invariantive properties of k-dimensional linear systems of real quadratic forms over $\boldsymbol{R}^n$ (or quadratic forms over $\boldsymbol{R}^n$ with values in $\boldsymbol{R}^k$, $2 \leqq k \leqq \frac{1}{2}n(n+1)$), because of the occurrence of such objects at least in the following contexts: in the second fundamental form of classical differential geometry for n-dimensional manifolds differentiably imbedded in an $(n+k)$-space, and in the study of certain matrix differential equations. For this reason there may be some interest in the following result concerning pencils (i.e., 2-dimensional systems) of quadratic forms.

2. THEOREM 1.[2] *Let $P(\mathrm{x})$, $Q(\mathrm{x})$ be two real quadratic forms over $\boldsymbol{R}^n$ ($\mathrm{x} \in \boldsymbol{R}^n$), with $3 \leqq n < \infty$. If the only $\mathrm{x} \in \boldsymbol{R}^n$ satisfying $P(\mathrm{x}) = Q(\mathrm{x}) = 0$ is $\mathrm{x} = 0$, then there exists a linear combination of P and Q with real coefficients, that is a positive definite quadratic form over $\boldsymbol{R}^n$.*

PROOF. From the hypothesis, the mapping assigning to each $\mathrm{x} \in \boldsymbol{R}^n$, other than 0, the pair $(P(\mathrm{x}), Q(\mathrm{x})) \in \boldsymbol{R}^2 - \{(0, 0)\}$, being homogeneous (of weight 2), defines by passage to quotients a continuous mapping f: $\boldsymbol{P}^{n-1}(\boldsymbol{R}) \to \boldsymbol{P}^1(\boldsymbol{R})$, where $\boldsymbol{P}^k(\boldsymbol{R})$ denotes the k-dimensional real projective space.

Since $n \geqq 3$, the fundamental group $\pi_1(\boldsymbol{P}^{n-1}(\boldsymbol{R}))$ is the cyclic group $\boldsymbol{Z}_2$ with two elements, while $\pi_1(\boldsymbol{P}^1(\boldsymbol{R})) \cong \boldsymbol{Z}$, the infinite cyclic group. Hence the induced homomorphism $f_*: \pi_1(\boldsymbol{P}^{n-1}(\boldsymbol{R})) \to \pi_1(\boldsymbol{P}^1(\boldsymbol{R}))$ is trivial (a catamorphism, should one say?). This fact implies that one can factor the map f into the commutative diagram

$$\begin{tikzcd} & \boldsymbol{R} \arrow[dr, "h"] & \\ \boldsymbol{P}^{n-1}(\boldsymbol{R}) \arrow[ur, dashed, "g"] \arrow[rr, "f"] & & \boldsymbol{P}^1(\boldsymbol{R}) \end{tikzcd},$$

where h is the universal covering map of the real line $\boldsymbol{R}$ onto $\boldsymbol{P}^1(\boldsymbol{R})$. Clearly, since $n < \infty$, the image $g(\boldsymbol{P}^{n-1}(\boldsymbol{R})) \subset \boldsymbol{R}$ is a bounded, closed segment I: denote by p the restriction of the covering map h to I, so that $f = pg$.

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[2] Cf. Remark 2 at the end of this paper.

Concerning the map $p\colon I \to \boldsymbol{P}^1(\boldsymbol{R})$, we consider the two possibilities: either $p(I) = \boldsymbol{P}^1(\boldsymbol{R})$ or $p(I) \subsetneq \boldsymbol{P}^1(\boldsymbol{R})$. In the former case, there exists at least one point $\langle(a, b)\rangle \in \boldsymbol{P}^1(\boldsymbol{R})$ (notation: $\langle(a, b)\rangle$ signifies the homogeneous equivalence class in $\boldsymbol{P}^1(\boldsymbol{R})$ defined by $(a, b) \in \boldsymbol{R}^2 - \{(0, 0)\}$), such that $p^{-1}(\langle(a, b)\rangle)$ contains at least two distinct points $t, t' \in I$: such being the case, the two subsets $g^{-1}(t)$, $g^{-1}(t')$ of $\boldsymbol{P}^{n-1}(\boldsymbol{R})$ are nonempty, disjunct, relatively open and closed subsets of the real, $(n-2)$-dimensional, projective quadric in $\boldsymbol{P}^{n-1}(\boldsymbol{R})$ defined by the homogeneous equation $bP(\mathbf{x}) - aQ(\mathbf{x}) = 0$. But it is well known that real projective quadric hypersurfaces, other than zero-dimensional ones, are either empty or connected point sets: hence this case cannot occur. We have shown thus that there exists a nonempty, open arc J in the circle $\boldsymbol{P}^1(\boldsymbol{R})$, that is the complement of the image $p(I) = f(\boldsymbol{P}^{n-1}(\boldsymbol{R}))$. If (a, b) denotes the homogeneous coordinates of any point in J, then, for every nonzero vector $\mathbf{x} \in \boldsymbol{R}^n$, the quadratic form $bP(\mathbf{x}) - aQ(\mathbf{x})$ takes a value different from zero; the quadratic form $bP - aQ$ is therefore either positive or negative definite (in which case $aQ - bP$ is positive). This completes the proof of the theorem.

3. A generalization of Theorem 1 to infinite-dimensional spaces leads only to a weaker conclusion. In fact, the two following quadratic forms in the Hilbert space of real sequences $(x_i)_{i \geq 1} = \mathbf{x}$ with summable squares, $P(\mathbf{x}) = \sum_{k=2}^{\infty} (1/k)x_k^2$ and $Q(\mathbf{x}) = x_1^2 - \sum_{i=2}^{\infty} x_i^2$ clearly satisfies the hypotheses of Theorem 1 (except $n < \infty$), but obviously no linear combination of P and Q is positive definite. This counterexample can be easily modified to work in any other type of infinite-dimensional real vector space (with or without norm). However, the following statement remains valid.

THEOREM 2. *Let $P(\mathbf{x})$ and $Q(\mathbf{x})$ be two quadratic forms in an infinite-dimensional, real vector space $\boldsymbol{V}$, such that $P(\mathbf{x}) = Q(\mathbf{x}) = 0$ implies $\mathbf{x} = 0$. Then there exists a linear combination of P and Q that is positive semi-definite in $\boldsymbol{V}$; if $\boldsymbol{W}$ is the subspace of $\boldsymbol{V}$ consisting of the vectors where this linear combination vanishes, then the restrictions of P and Q to $\boldsymbol{W}$ are both constant multiples, not both zero, of a common, positive definite quadratic form on $\boldsymbol{W}$.*

PROOF. One follows all the steps of the proof of Theorem 1; the only exception is that the interval $I = g(\boldsymbol{P}_V(\boldsymbol{R})) \subset \boldsymbol{R}$ is, in general, not compact, since the projective space $\boldsymbol{P}_V(\boldsymbol{R})$ associated with an infinite-dimensional vector space $\boldsymbol{V}$ is not compact. On the other hand, the same argument as in Theorem 1 shows that the restriction p of the covering map $h\colon \boldsymbol{R} \to \boldsymbol{P}^1(\boldsymbol{R})$ to the interval I is a one-to-one map.

The linear combination $\pm(bP-aQ)$ with the desired property may be chosen for any (a, b) equal to homogeneous coordinates of any point $\langle(a, b)\rangle$ in the (nonempty) complement in $\boldsymbol{P}^1(\boldsymbol{R})$ of the open, proper subarc $p(\mathrm{Int}(I))$, where $\mathrm{Int}(I)$ denotes the interior of the interval I. This concludes the proof of the existence of a positive semidefinite linear combination of P and Q. The remaining conclusions follow trivially.

4. The statement of Theorems 1 and 2 is clearly false in the case of two-dimensional vector spaces, as shown by the example of the two binary quadratic forms $P(x, y)=x^2-y^2$, $Q(x, y)=2xy$. Similarly, corresponding statements fail in the case of k-dimensional linear systems of quadratic forms on $\boldsymbol{R}^n$ for $n\geqq 3$ and $3\leqq k<\frac{1}{2}n(n+1)$, even though the converse statements remain, in all cases, trivially valid. In fact, the linear system of quadratic forms with trace zero (referred to a suitable basis) form a maximal system, in the finite-dimensional case, containing no positive semidefinite forms; the system is precisely $(\frac{1}{2}n(n+1)-1)$-dimensional. From such a system, nevertheless, one can select, for $n\geqq 3$, three quadratic forms with no common zeros outside of the origin, for instance; $P_1(\mathbf{x})=x_1^2-x_2^2$, $P_2(\mathbf{x})=2x_1x_2$, $P_3(\mathbf{x})=x_1^2+x_2^2-(2/n-2)\sum_{i=3}^n x_i^2$.

REMARKS. 1. The whole content of Theorem 1 is an elementary property of the real number field. By the Tarski principle, the same theorem remains valid, if the real number field $\boldsymbol{R}$ is replaced by any other real-closed field. This furnishes one more example, in which topological methods are used to prove a statement of a purely algebraic nature. I do not know at present a purely algebraic proof.

2. Dr. Olga Taussky Todd has pointed out that, in essence, Theorem 1 is already known; it is equivalent to a theorem in Graeub's textbook [*Linear algebra*, Springer, 1958, pp. 158–163], stating that under the hypotheses of Theorem 1, the quadratic forms P and Q can be simultaneously diagonalized by a suitable choice of basis. The conclusion of Theorem 1 follows then from a trivial application of a result on linear inequalities [Stiemke, Math. Ann. **76** (1915), 340–342; also Annals of Mathematics Studies No. 24, *Contributions to the theory of games*, Princeton Univ. Press, Princeton, N. J.]; on the other hand, the simultaneous diagonalization of P and Q, given that they have a positive definite linear combination, is classical.

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