DIFFERENTIAL AND INTEGRAL INEQUALITIES

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This note presents new proofs for some important inequalities [1]. The assumptions on positivity or monotony of the various functions are weaker than those in [1] or in the original references (see [1]) and yet the method seems astonishingly elementary. We set $u' = du/dt$, and $t \geqq 0$; the reversal of inequalities for $t<0$ can be read off from the proofs. Since continuous functions are dense in the integral norm, we suppose all functions continuous.

Let $Tu = u' - a(t)b(u)$, where $b>0$. Then

$$(1) \qquad Tu \leqq 0 \leqq Tv \quad \text{and} \quad u(0) \leqq v(0) \Rightarrow u \leqq v.$$

For proof define $\tilde{u} = B(u)$ and $\tilde{v} = B(v)$, where $B(y) = \int_\delta^y [b(s)]^{-1} ds$. Then

$$b(u)[\tilde{u}' - a(t)] \leqq 0 \leqq b(v)[\tilde{v}' - a(t)],$$

therefore $\tilde{u}' \leqq a(t) \leqq \tilde{v}'$, and therefore $\tilde{u} - \tilde{v}$ is nonincreasing. But $\tilde{u}(0) \leqq \tilde{v}(0)$. Hence $\tilde{u} \leqq \tilde{v}$, and consequently $u \leqq v$. The proof shows that the differential inequality is needed only at points where $u>v$.

The equation $Tv=0$, $v(0)=\delta$ can be solved by inspection, to give:

$$(2) \quad Tu \leqq 0 \text{ and } u(0) \leqq \delta \Rightarrow u \leqq B^{-1}[A(t)], \text{ where } A(t) = \int_0^t a(s)ds.$$

In this paragraph only, let u and v be vectors, with any convenient norm, and let $\tilde{T}u = u' - f(t, u)$, where f is a vector. Suppose

$$\|f(t, u) - f(t, v)\| \leqq a(t)b(w) \quad \text{where} \quad w = \|u - v\|.$$

If $\tilde{T}u = \tilde{T}v$ and $\|u - v\| = \delta$ for $t=0$, then the fact that $|\, \|w\|' \,| \leqq \|w'\|$ gives

$$pw' \leqq \|u' - v'\| \leqq a(t)b(w), \qquad p = + \text{ or } p = -.$$

Without loss of generality take b even, so $b(pw) = b(w)$. Let $v = v^p$ satisfy $(pv)' = a(t)b(pv)$, $v(0) = \delta$. Then $T(pw) \leqq 0 = T(pv)$, $pw(0) = pv(0)$. Hence $pw \leqq pv^p$, which is to say, $v^- \leqq w \leqq v^+$. Since v^p satisfies $v' = pa(t)b(v)$, we use pA in (2) and get:

$$(3) \qquad B^{-1}[-A(t)] \leqq \|u - v\| \leqq B^{-1}[A(t)].$$

This sharpens the theorems of Bihari and Langenhop.

Received by the editors April 26, 1963.

If $a \geqq 0$ we can allow $b \geqq 0$ instead of $b > 0$: Replace b by $b+h$, where h is a positive constant. For the corresponding operator T_h we have $T_h u \leqq Tu \leqq 0$, hence $u \leqq v_h$. The resulting inequality for $h \to 0$ depends on the convergence or divergence of the integral defining B at the zeros of b. Since equation (2) for v_h can be written

$$Tu \leqq 0 \text{ and } u(0) \leqq \delta \Rightarrow u \leqq y, \text{ where } \int_\delta^v \frac{ds}{b(s)+h} \geqq \int_0^t a(s)ds,$$

the behavior as $h \to 0$ is analyzed with ease.

When not only $a \geqq 0$ but b is monotone nondecreasing as a function of its argument, u, we can analyze the integral inequality

$$w \leqq \delta + \int_0^t a(s)b[w(s)]ds.$$

If the right side is called u then $u' = a(t)b(w) \leqq a(t)b(u)$, so that $Tu \leqq 0$, $u(0) = \delta$. Therefore the estimate (2) holds for u, and *a fortiori* for w. This sharpens another theorem of Bihari. The choice $\delta = a = 1$, $b(w) = 4(w+1)^{-1}$, $w = t^2$, $t = 2$, shows that the monotony of b cannot be dropped here, although it was superfluous in (1).

Reference

1. E. F. Beckenbach and Richard Bellman, *Inequalities*, pp. 133–136, Springer, Berlin, 1961.

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