THE ANNIHILATOR OF A KNOT MODULE[1]

R. H. CROWELL

1. **Introduction.** Let Zt be the integral group ring of an infinite cyclic multiplicative group generated by t. If $K \subset S^3$ is a tame knot with group $G = \pi_1(S^3 - K)$ and commutator subgroup G', then G/G' is infinite cyclic, and its integral group ring is therefore isomorphic to Zt. An important set of invariants of K, discussed in [3], are the knot polynomials $\Delta_1, \Delta_2, \cdots$, which belong to Zt and satisfy $\Delta_{k+1} | \Delta_k$ and $|\Delta_k(1)| = 1$. The abelianized commutator subgroup G'/G'', if written additively, is a Zt-module: Let $\nu: G \to G/G'$ and $\mu: G' \to G'/G''$ be the abelianizing maps, and $t = \nu g$ a generator of G/G'. The action of t on an arbitrary $\mu k \in G'/G''$ is defined by $t \cdot \mu k = \mu(gkg^{-1})$. If a column is deleted from the Alexander matrix of any Wirtinger presentation of G, it follows directly from [4, §7] that the resulting square matrix is a relation matrix for G'/G''. In this paper we show that *the annihilating ideal of the Zt-module G'/G'' is principal and generated by Δ_1/Δ_2.*

2. **Noetherian modules.** Let R be a commutative noetherian ring, and A a finitely generated R-module. The set of all $\alpha \in R$ such that $\alpha \cdot a = 0$ for all $a \in A$ is an ideal, which we call the annihilator of A and denote by $\mathfrak{a}$. Since any finitely generated module over R is noetherian [2, p. 15], there exists an exact sequence

$$X_1 \xrightarrow{d} X_0 \xrightarrow{e} X \to 0 \tag{1}$$

of R-morphisms, called a *presentation sequence*, in which X_0 and X_1 are free R-modules with bases $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_m)$, respectively. The matrix (α_{ij}) defined by

$$dy_i = \sum_{j=1}^{n} \alpha_{ij} \cdot x_j, \qquad i = 1, \cdots, m,$$

is a *relation matrix* for A. For each positive integer k, the kth *elementary ideal*[2] of (α_{ij}) is the ideal $\mathfrak{e}_k$ generated by the determinants of all $(n-k+1) \times (n-k+1)$ submatrices of (α_{ij}). We adopt the convention that $\mathfrak{e}_k = 0$ if $m < (n-k+1)$, and $\mathfrak{e}_k = R$ if $(n-k+1) < 1$. Obviously,

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[2] We have numbered the elementary ideals so that $\mathfrak{e}_k$ corresponds to Δ_k in the application to knot modules. This numbering differs from both [3] and [7].

$$\mathfrak{e}_k \subset \mathfrak{e}_{k+1}.$$

It is proved in [7, p. 90] that *the elementary ideals depend only on* A. Moreover,

(2.1) $\mathfrak{e}_1 \subset \mathfrak{a}$,

(2.2) $\sqrt{\mathfrak{e}_1} = \sqrt{\mathfrak{a}}$.

A proof of (2.1), is given in [7, p. 93]. To establish (2.2), we first prove that *if* R *is an integral domain and* $\mathfrak{e}_1 = 0$, *then* $\mathfrak{a} = 0$. Let K be the quotient field of R, and apply $K \otimes_R$ to a presentation sequence (1). Then $1 \otimes d$ is a linear transformation of finite dimensional vector spaces, and, because $\mathfrak{e}_1 = 0$, its rank is $\leqq n-1$. Hence $K \otimes_R A \neq 0$, and it therefore contains an element $\kappa \otimes_R a \neq 0$. It follows that zero is the only element in R that can annihilate a, and so $\mathfrak{a} = 0$.

To prove (2.2), it suffices to show that every prime ideal $\mathfrak{p}$ that contains $\mathfrak{e}_1$ also contains $\mathfrak{a}$. Consider $\phi: R \to R/\mathfrak{p}$, and apply $R/\mathfrak{p} \otimes_R$ to a presentation sequence (1). In the resulting sequence the matrix of $1 \otimes d$ is $(\phi\alpha_{ij})$ and is a relation matrix for $R/\mathfrak{p} \otimes_R A$. Its 1st elementary ideal is $\phi\mathfrak{e}_1 = 0$. Since $R/\mathfrak{p}$ is an integral domain, the above lemma implies that the annihilator of $R/\mathfrak{p} \otimes_R A$, which contains $\phi\mathfrak{a}$, is zero. Hence $\mathfrak{a} \subset \mathfrak{p}$, and the proof is complete.

We define the *deficiency* of A to be the maximum of all numbers $n-m$ such that there exists an $m \times n$ relation matrix for A. Notice that the deficiency of A cannot exceed the number of trivial elementary ideals of A.

(2.3) *Suppose that* R *is an integral domain and that* A *has nonnegative deficiency. If* $\pi \cdot a = 0$ *for some prime* $\pi \in R$ *and nonzero* $a \in A$, *then* $\mathfrak{e}_1 \subset (\pi)$.

PROOF. Since any number of zero rows can be adjoined to a relation matrix, we may assume that A has a presentation sequence (1) for which $m = n$. Choose $\alpha_1, \cdots, \alpha_n \in R$ so that $e(\sum_{j=1}^n \alpha_j \cdot x_j) = a$. Since $\pi \cdot a = 0$,

$$\sum_{j=1}^n \pi\alpha_j \cdot x_j = d \sum_{i=1}^n \beta_i \cdot y_i = \sum_{j=1}^n \left(\sum_{i=1}^n \beta_i \alpha_{ij} \right) \cdot x_j.$$

Hence

$$\pi\alpha_j = \sum_{i=1}^n \beta_i \alpha_{ij}, \qquad j = 1, \cdots, n.$$

Writing $\phi: R \to R/(\pi)$, we obtain

$$\phi(\pi\alpha_j) = 0 = \sum_{i=1}^n \phi\beta_i \phi\alpha_{ij}, \qquad j = 1, \cdots, n.$$

It follows that either $\phi\beta_1 = \cdots = \phi\beta_n = 0$ or $\det(\phi\alpha_{ij}) = \phi \det(\alpha_{ij}) = 0$. Since R is an integral domain, the first alternative implies $a = 0$ and is therefore false. We conclude that $\phi\mathfrak{e}_1 = \phi(\det(\alpha_{ij})) = (\phi \det(\alpha_{ij})) = 0$. Hence $\mathfrak{e}_1 \subset (\pi)$, and the proof is complete.

The preceding result is false without the requirement that A have nonnegative deficiency. Consider the exact sequence $0 \to (2, t) \to Z[t] \to Z_2 \to 0$, and take the polynomial ring $Z[t]$ for R, Z_2 for A, 2 for π, and $1 \in Z_2$ for a. Then $\pi \cdot a = 0$ and $\mathfrak{e}_1 = \mathfrak{a} = (2, t)$, which is not contained in (π).

3. **Knot modules.** Let A be a finitely generated Zt-module. Since Zt, besides being noetherian, is a unique factorization domain, any ideal $\mathfrak{b}$ is contained in a smallest principal ideal $\mathfrak{b}^*$, equal to the intersection of all principal ideals containing $\mathfrak{b}$, see [**3**, Chapter VIII, §2]. The kth *knot polynomial* Δ_k is the generator, determined to within a unit multiple, of $\mathfrak{e}_k^*$. Clearly $\Delta_{k+1} | \Delta_k$. A nonzero element $\alpha = \sum n_k t^k$ in Zt is *primitive* if its coefficients $\{n_k\}$ are relatively prime.

Let Q be the field of rational numbers. Then $Qt \cong Q \otimes_Z Zt$, the group ring with rational coefficients, is a principal ideal domain which contains Zt.

(3.1) *Let $\mathfrak{b}$ be an ideal in Zt. If β generates $\mathfrak{b}^*$, then β also generates the smallest ideal in Qt that contains $\mathfrak{b}$.*

PROOF. We may assume $\mathfrak{b} \neq 0$. Clearly $\mathfrak{b} \subset \mathfrak{b}^* \subset (\beta)_{Qt}$. Conversely, let $\mathfrak{c}$ be any ideal in Qt containing $\mathfrak{b}$, and choose a generator γ which is in Zt and is primitive. For any $\alpha \in \mathfrak{b}$, we have $\alpha = \gamma\delta$ for $\delta \in Qt$. Gauss' lemma [**1**, p. 79] that the product of two primitive elements is primitive implies that in fact $\delta \in Zt$, and so $\mathfrak{b} \subset (\gamma)_{Zt}$. Since $\mathfrak{b}^*$ is minimal, $\mathfrak{b}^* = (\beta)_{Zt} \subset (\gamma)_{Zt}$. This implies $(\beta)_{Qt} \subset (\gamma)_{Qt} = \mathfrak{c}$ and completes the proof.

Consider a presentation sequence (1) for A with relation matrix (α_{ij}). Applying $Qt \otimes_{Zt}$, we obtain a presentation sequence for $Qt \otimes_{Zt} A$ with the same relation matrix except that the entries α_{ij} are now considered to be in Qt. It follows by (3.1) that Δ_k generates the kth elementary ideal of the Qt-module $Qt \otimes_{Zt} A$. Hence the structure theorem for finitely generated modules over a principal ideal domain implies that:

(3.2) *The annihilator of $Qt \otimes_{Zt} A$ is generated by Δ_1/Δ_2.*

Since $A \cong Zt \otimes_{Zt} A$ under $a \to 1 \otimes_{Zt} a$, we obtain

$$Q \otimes_Z A \cong Q \otimes_Z (Zt \otimes_{Zt} A) \cong (Q \otimes_Z Zt) \otimes_{Zt} A \cong Qt \otimes_{Zt} A$$

under the map g defined by $g(q \otimes_Z a) = q \otimes_{Zt} a$. We therefore get the consistent diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & Q \otimes_Z A \arrow[d, "g"] \arrow[dl, phantom, "\cong"] \\ & Qt \otimes_{Zt} A \end{tikzcd} \tag{2}$$

where $fa = 1 \otimes_Z a$ and $ha = 1 \otimes_{Zt} a$.

(3.3) *Let A be torsion free as a Z-module. Then Δ_1 is primitive if it is nonzero, and the annihilator $\mathfrak{a} \subset Zt$ is the principal ideal generated by Δ_1/Δ_2.*

PROOF. If $\Delta_1 = 0$, then $\mathfrak{e}_1 = 0$ and by (2.2) also $\mathfrak{a} = 0$. So we assume $\Delta_1 \neq 0$, and consequently $\mathfrak{e}_1$ and $\mathfrak{a}$ are also nonzero. If Δ_1 is not primitive, it is divisible by some prime integer p. Hence $\mathfrak{e}_1 \subset (\Delta_1) \subset (p)$, and the latter is a prime ideal (Gauss' lemma). Hence $\mathfrak{a} \subset (p)$ by (2.2). Choose $\alpha \neq 0$ in $\mathfrak{a}$ and write $\alpha = p^k \beta$, where $p \nmid \beta$. Then $\beta \notin \mathfrak{a}$, so there exists $a \in A$ with $\beta \cdot a \neq 0$. Some one of the elements $p^j \beta \cdot a$, $j = 0, \cdots, k-1$, is nonzero and of order p. This proves the first assertion.

Set $\delta = \Delta_1/\Delta_2$. Since A is a torsion free abelian group, the map f in the diagram (2), and consequently h as well, is a monomorphism [2, p. 130]. For any $a \in A$, we have $0 = \delta \cdot ha = \delta \cdot (1 \otimes_{Zt} a) = 1 \otimes_{Zt} \delta \cdot a = h(\delta \cdot a)$, and so $\delta \cdot a = 0$. Hence $\delta \in \mathfrak{a}$. Conversely, suppose that $\alpha \in \mathfrak{a}$. Then for any $\beta \in Qt$ and $a \in A$, we have $\alpha \cdot (\beta \otimes_{Zt} a) = (\beta \otimes_{Zt} \alpha \cdot a) = 0$, and so by (3.2) we have $\alpha = \gamma\delta$ for some $\gamma \in Qt$. But δ is primitive because Δ_1 is, so Gauss' lemma again implies $\gamma \in Zt$. We conclude that $\mathfrak{a} = (\delta)_{Zt}$, and the proof is complete.

A partial converse to (3.3) is:

(3.4) *If A has nonnegative deficiency and $\Delta_1 \neq 0$ is primitive, then A is a torsion free abelian group.*[3]

PROOF. In fact, the deficiency must be zero, and $\mathfrak{e}_1 = (\Delta_1)$. Suppose A contains a nonzero element of prime order p. As we have noted above, p generates a prime ideal of Zt. It follows by (2.3) that $(\Delta_1) = \mathfrak{e}_1 \subset (p)$, and so Δ_1 is not primitive.

The knot module G'/G'' in the introduction has deficiency zero, and its 1st polynomial is nonzero and primitive because $|\Delta_1(1)| = 1$. It follows from (3.4) that G'/G'' is torsion free as a Z-module and from (3.3) that, as a Zt-module, its annihilator is (Δ_1/Δ_2).

The square knot, for example, has polynomials $\Delta_1 = (t^2 - t + 1)^2$ and $\Delta_2 = t^2 - t + 1$; so the annihilator $\mathfrak{a}$ is the prime ideal generated by $t^2 - t + 1$. For the knot 8_{20n} [6, pp. 41, 71], however, $\Delta_1 = (t^2 - t + 1)^2$ and $\Delta_2 = 1$, and $\mathfrak{a}$ is therefore the primary ideal generated by $(t^2 - t + 1)^2$.

[3] This is also proved in [5].

BIBLIOGRAPHY

1. G. Birkhoff and S. MacLane, *A survey of modern algebra*, rev. ed., Macmillan, New York, 1953.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

3. R. Crowell and R. Fox, *Introduction to knot theory*, Ginn, Boston, 1963.

4. R. Crowell, *Corresponding group and module sequences*, Nagoya Math. J. **19** (1961), 27–40.

5. ———, *The group G'/G'' of a knot group G*, Duke Math. J. **30** (1963), 349–354.

6. K. Reidemeister, *Knotentheorie*, Chelsea, New York, 1948.

7. H. Zassenhaus, *The theory of groups*, Chelsea, New York, 1949.

DARTMOUTH COLLEGE

NOTE ON POINTWISE PERIODIC SEMIGROUPS

ANNE L. HUDSON[1]

An element x in a semigroup S is said to be periodic if there exists a positive integer n such that $x^{n+1}=x$, and the least such n, $p(x)$, is the period of x. S is pointwise periodic if each x in S is periodic. In [4], A. D. Wallace asks the following question concerning pointwise periodic topological semigroups.

Problem 3: If S is a pointwise periodic semigroup and is topologically an n-cell, is it possible that $S\backslash E$ is nonempty and $p(x)>2$ and constant on $S\backslash E$?

It will be shown that in a slightly more general situation than that of the above problem, it necessarily follows that $p(x)=2$ on $S\backslash E$.

The following notation will be used throughout this paper. For a semigroup S, $E=\{x: x\in S, x^2=x\}$ and for $e\in E$, $H(e)$ is the maximal subgroup of S containing the idempotent e. $H=\bigcup\{H(e): e\in E\}$ and functions y and θ are defined as in [5], that is, for $x\in H$, $y(x)$ is the idempotent of the unique maximal subgroup to which x belongs and $\theta(x)$ is the inverse of x in this group.

The following theorem will be proved:

THEOREM. *Let S be a compact semigroup with the properties*:

(1) $S=H$,

(2) *for $e\in E$, $H(e)$ is totally disconnected,*

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