THE ANNIHILATOR OF A KNOT MODULE\textsuperscript{1}

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1. Introduction. Let $\mathbb{Z}t$ be the integral group ring of an infinite cyclic multiplicative group generated by $t$. If $K \subset S^3$ is a tame knot with group $G = \pi_1(S^3 - K)$ and commutator subgroup $G'$, then $G/G'$ is infinite cyclic, and its integral group ring is therefore isomorphic to $\mathbb{Z}t$. An important set of invariants of $K$, discussed in [3], are the knot polynomials $\Delta_1, \Delta_2, \ldots$, which belong to $\mathbb{Z}t$ and satisfy $\Delta_{k+1} | \Delta_k$ and $|\Delta_k(1)| = 1$. The abelianized commutator subgroup $G'/G''$, if written additively, is a $\mathbb{Z}t$-module: Let $\nu: G \to G/G'$ and $\mu: G' \to G'/G''$ be the abelianizing maps, and $t = \nu g$ a generator of $G/G'$. The action of $t$ on an arbitrary $kg \in G'/G''$ is defined by $t \cdot kg = g(g^{-1})$. If a column is deleted from the Alexander matrix of any Wirtinger presentation of $G$, it follows directly from [4, §7] that the resulting square matrix is a relation matrix for $G'/G''$. In this paper we show that the annihilating ideal of the $\mathbb{Z}t$-module $G'/G''$ is principal and generated by $\Delta_1/\Delta_2$.

2. Noetherian modules. Let $R$ be a commutative noetherian ring, and $A$ a finitely generated $R$-module. The set of all $a \in R$ such that $a \cdot a = 0$ for all $a \in A$ is an ideal, which we call the annihilator of $A$ and denote by $\text{ann}(A)$. Since any finitely generated module over $R$ is noetherian [2, p. 15], there exists an exact sequence

\[ X_1 \xrightarrow{d} X_0 \xrightarrow{e} X \to 0 \]

of $R$-morphisms, called a presentation sequence, in which $X_0$ and $X_1$ are free $R$-modules with bases $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$, respectively. The matrix $(\alpha_{ij})$ defined by

\[ dy_i = \sum_{j=1}^{n} \alpha_{ij} \cdot x_j, \quad i = 1, \ldots, m, \]

is a relation matrix for $A$. For each positive integer $k$, the $k$th elementary ideal\textsuperscript{2} of $(\alpha_{ij})$ is the ideal $\epsilon_k$ generated by the determinants of all $(n-k+1) \times (n-k+1)$ submatrices of $(\alpha_{ij})$. We adopt the convention that $\epsilon_k = 0$ if $m < (n-k+1)$, and $\epsilon_k = R$ if $(n-k+1) < 1$. Obviously,

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\textsuperscript{2} We have numbered the elementary ideals so that $\epsilon_k$ corresponds to $\Delta_k$ in the application to knot modules. This numbering differs from both [3] and [7].
It is proved in [7, p. 90] that the elementary ideals depend only on \( A \).
Moreover,

\begin{align}
(2.1) & \quad \epsilon_1 \subseteq \alpha, \\
(2.2) & \quad \sqrt{\epsilon_1} = \sqrt{\alpha}.
\end{align}

A proof of (2.1), is given in [7, p. 93]. To establish (2.2), we first prove that if \( R \) is an integral domain and \( \epsilon_1 = 0 \), then \( \alpha = 0 \). Let \( K \) be the quotient field of \( R \), and apply \( K \otimes_R \) to a presentation sequence (1). Then \( 1 \otimes d \) is a linear transformation of finite dimensional vector spaces, and, because \( \epsilon_1 = 0 \), its rank is \( \leq n - 1 \). Hence \( K \otimes_R \alpha \neq 0 \), and it therefore contains an element \( k \otimes_R \alpha \neq 0 \). It follows that zero is the only element in \( R \) that can annihilate \( \alpha \), and so \( \alpha = 0 \).

To prove (2.2), it suffices to show that every prime ideal \( p \) that contains \( \epsilon_1 \) also contains \( \alpha \). Consider \( \phi: R \rightarrow R/p \), and apply \( R/p \otimes_R \) to a presentation sequence (1). In the resulting sequence the matrix of \( 1 \otimes d \) is \( (\phi \alpha_{ij}) \) and is a relation matrix for \( R/p \otimes_R \alpha \). Its 1st elementary ideal is \( \phi \epsilon_1 = 0 \). Since \( R/p \) is an integral domain, the above lemma implies that the annihilator of \( R/p \otimes_R \alpha \), which contains \( \phi \alpha \), is zero. Hence \( \alpha \subseteq p \), and the proof is complete.

We define the deficiency of \( \alpha \) to be the maximum of all numbers \( n - m \) such that there exists an \( m \times n \) relation matrix for \( A \). Notice that the deficiency of \( A \) cannot exceed the number of trivial elementary ideals of \( A \).

\begin{align}
(2.3) & \quad \text{Suppose that } R \text{ is an integral domain and that } \alpha \text{ has nonnegative deficiency. If } \pi \cdot \alpha = 0 \text{ for some prime } \pi \in R \text{ and nonzero } \alpha \in \alpha, \text{ then } \epsilon_1 \subseteq (\pi).
\end{align}

**Proof.** Since any number of zero rows can be adjoined to a relation matrix, we may assume that \( A \) has a presentation sequence (1) for which \( m = n \). Choose \( \alpha_1, \ldots, \alpha_n \in R \) so that \( e(\sum_{j=1}^{n} \alpha_j x_j) = a \). Since \( \pi \cdot a = 0 \),

\[
\sum_{i=1}^{n} \pi \alpha_i : x_j = d \sum_{i=1}^{n} \beta_i : y_i = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \beta_i \alpha_{ij} \right) : x_j.
\]

Hence

\[
\pi \alpha_j = \sum_{i=1}^{n} \beta_i \alpha_{ij}, \quad j = 1, \ldots, n.
\]

Writing \( \phi: R \rightarrow R/(\pi) \), we obtain

\[
\phi(\pi \alpha_j) = 0 = \sum_{i=1}^{n} \phi \beta_i \phi \alpha_{ij}, \quad j = 1, \ldots, n.
\]
It follows that either $\phi_{\beta_1} = \cdots = \phi_{\beta_n} = 0$ or $\det(\phi_{\alpha_{ij}}) = \phi \det(\alpha_{ij}) = 0$. Since $R$ is an integral domain, the first alternative implies $a = 0$ and is therefore false. We conclude that $\phi_{\beta_1} = \phi(\det(\alpha_{ij})) = (\phi \det(\alpha_{ij})) = 0$. Hence $e_1 \subseteq \langle \pi \rangle$, and the proof is complete.

The preceding result is false without the requirement that $A$ have nonnegative deficiency. Consider the exact sequence $0 \to (2, i) \to \mathbb{Z}[i] \to \mathbb{Z}_2 \to 0$, and take the polynomial ring $\mathbb{Z}[i]$ for $R$, $\mathbb{Z}_2$ for $A$, 2 for $\pi$, and $1 \in \mathbb{Z}_2$ for $a$. Then $\pi \cdot a = 0$ and $e_1 = a = (2, i)$, which is not contained in $\langle \pi \rangle$.

3. Knot modules. Let $A$ be a finitely generated $\mathbb{Z}_t$-module. Since $\mathbb{Z}_t$, besides being noetherian, is a unique factorization domain, any ideal $b$ is contained in a smallest principal ideal $b^*$, equal to the intersection of all principal ideals containing $b$, see [3, Chapter VIII, §2]. The $k$th knot polynomial $\Delta_k$ is the generator, determined to within a unit multiple, of $e_k^*$. Clearly $\Delta_{k+1} \subseteq \Delta_k$. A nonzero element $\alpha = \sum n_k t^k$ in $\mathbb{Z}_t$ is primitive if its coefficients $\{n_k\}$ are relatively prime.

Let $Q$ be the field of rational numbers. Then $Qt \cong Q \otimes_{\mathbb{Z}_t} \mathbb{Z}_t$, the group ring with rational coefficients, is a principal ideal domain which contains $\mathbb{Z}_t$.

(3.1) Let $b$ be an ideal in $\mathbb{Z}_t$. If $\beta$ generates $b^*$, then $\beta$ also generates the smallest ideal in $Qt$ that contains $b$.

**Proof.** We may assume $b \neq 0$. Clearly $b \subseteq b^* \subseteq (\beta)_{Qt}$. Conversely, let $c$ be any ideal in $Qt$ containing $b$, and choose a generator $\gamma$ which is in $\mathbb{Z}_t$ and is primitive. For any $\alpha \in b$, we have $\alpha = \gamma \delta$ for $\delta \in Qt$. Gauss’ lemma [1, p. 79] that the product of two primitive elements is primitive implies that in fact $\delta \in \mathbb{Z}_t$, and so $b \subseteq \langle \gamma \rangle_{\mathbb{Z}_t}$. Since $b^*$ is minimal, $b^* = \langle \beta \rangle_{\mathbb{Z}_t} \subseteq \langle \gamma \rangle_{\mathbb{Z}_t}$. This implies $\langle \beta \rangle_{Qt} \subseteq \langle \gamma \rangle_{Qt} = c$ and completes the proof.

Consider a presentation sequence (1) for $A$ with relation matrix $(\alpha_{ij})$. Applying $Qt \otimes_{\mathbb{Z}_t} A$, we obtain a presentation sequence for $Qt \otimes_{\mathbb{Z}_t} A$ with the same relation matrix except that the entries $\alpha_{ij}$ are now considered to be in $Qt$. It follows by (3.1) that $\Delta_k$ generates the $k$th elementary ideal of the $Qt$-module $Qt \otimes_{\mathbb{Z}_t} A$. Hence the structure theorem for finitely generated modules over a principal ideal domain implies that:

(3.2) The annihilator of $Qt \otimes_{\mathbb{Z}_t} A$ is generated by $\Delta_1/\Delta_2$.

Since $A \cong \mathbb{Z}_t \otimes_{\mathbb{Z}_t} A$ under $a \to 1 \otimes_{\mathbb{Z}_t} a$, we obtain

$$Q \otimes_{\mathbb{Z}_t} A \cong (Q \otimes_{\mathbb{Z}_t} (\mathbb{Z}_t \otimes_{\mathbb{Z}_t} A)) \cong (Q \otimes_{\mathbb{Z}_t} \mathbb{Z}_t) \otimes_{\mathbb{Z}_t} A \cong Qt \otimes_{\mathbb{Z}_t} A$$

under the map $g$ defined by $g(q \otimes_{\mathbb{Z}_t} a) = q \otimes_{\mathbb{Z}_t} a$. We therefore get the consistent diagram.

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A -- Q \otimes \mathbb{Z} A
\downarrow h 
\cong 
\downarrow g
Ql \otimes \mathbb{Z}_l A

where \(fa = 1 \otimes \mathbb{Z} a\) and \(ha = 1 \otimes \mathbb{Z}_l a\).

(3.3) Let \(A\) be torsion free as a \(\mathbb{Z}\)-module. Then \(\Delta_1\) is primitive if it is nonzero, and the annihilator \(a \subset \mathbb{Z}_l\) is the principal ideal generated by \(\Delta_1/\Delta_2\).

**Proof.** If \(\Delta_1 = 0\), then \(e_1 = 0\) and by (2.2) also \(a = 0\). So we assume \(\Delta_1 \neq 0\), and consequently \(e_1\) and \(a\) are also nonzero. If \(\Delta_1\) is not primitive, it is divisible by some prime integer \(p\). Hence \(e_1 \subset (\Delta_1) \subset (p)\), and the latter is a prime ideal (Gauss' lemma). Hence \(a \subset (p)\) by (2.2). Choose \(\alpha \neq 0\) in \(a\) and write \(\alpha = p^k \beta\), where \(p \not| \beta\). Then \(\beta \not\subset a\), so there exists \(a \subset A\) with \(\beta \cdot a \neq 0\). Some one of the elements \(p^j \beta \cdot a, j = 0, \cdots, k - 1\), is nonzero and of order \(p\). This proves the first assertion.

Set \(\delta = \Delta_1/\Delta_2\). Since \(A\) is a torsion free abelian group, the map \(f\) in the diagram (2), and consequently \(h\) as well, is a monomorphism [2, p. 130]. For any \(a \subset A\), we have \(0 = \delta \cdot ha = \delta \cdot (1 \otimes \mathbb{Z}_l a) = 1 \otimes \mathbb{Z}_l \delta \cdot a = h(\delta \cdot a)\), and so \(\delta \cdot a = 0\). Hence \(\delta \subset a\). Conversely, suppose that \(\alpha \subset a\). Then for any \(\beta \subset Ql\) and \(a \subset A\), we have \(\alpha \cdot (\beta \otimes \mathbb{Z}_l a) = (\beta \otimes \mathbb{Z}_l \alpha \cdot a) = 0\), and so by (3.2) we have \(\alpha = \gamma \delta\) for some \(\gamma \subset Ql\). But \(\delta\) is primitive because \(\Delta_1\) is, so Gauss' lemma again implies \(\gamma \subset \mathbb{Z}_l\). We conclude that \(a = (\delta) \subset Zl\), and the proof is complete.

A partial converse to (3.3) is:

(3.4) If \(A\) has nonnegative deficiency and \(\Delta_1 \neq 0\) is primitive, then \(A\) is a torsion free abelian group.

**Proof.** In fact, the deficiency must be zero, and \(e_1 = (\Delta_1)\). Suppose \(A\) contains a nonzero element of prime order \(p\). As we have noted above, \(p\) generates a prime ideal of \(\mathbb{Z}_l\). It follows by (2.3) that \((\Delta_1) = e_1 \subset (\mathbb{Z}_l)\), and so \(\Delta_1\) is not primitive.

The knot module \(G'/G''\) in the introduction has deficiency zero, and its 1st polynomial is nonzero and primitive because \(|\Delta_1(1)| = 1\). It follows from (3.4) that \(G'/G''\) is torsion free as a \(\mathbb{Z}\)-module and from (3.3) that, as a \(\mathbb{Z}_l\)-module, its annihilator is \((\Delta_1/\Delta_2)\).

The square knot, for example, has polynomials \(\Delta_1 = (t^2 - t + 1)^2\) and \(\Delta_2 = t^2 - t + 1\); so the annihilator \(a\) is the prime ideal generated by \(t^2 - t + 1\). For the knot \(8_{20a}\) [6, pp. 41, 71], however, \(\Delta_1 = (t^2 - t + 1)^2\) and \(\Delta_2 = 1\), and \(a\) is therefore the primary ideal generated by \((t^2 - t + 1)^2\).

\(^3\) This is also proved in [5].
Bibliography


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NOTE ON POINTWISE PERIODIC SEMIGROUPS

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An element x in a semigroup S is said to be periodic if there exists a positive integer n such that x^{n+1} = x, and the least such n, p(x), is the period of x. S is pointwise periodic if each x in S is periodic. In [4], A. D. Wallace asks the following question concerning pointwise periodic topological semigroups.

**Problem 3:** If S is a pointwise periodic semigroup and is topologically an n-cell, is it possible that S\E is nonempty and p(x) > 2 and constant on S\E?

It will be shown that in a slightly more general situation than that of the above problem, it necessarily follows that p(x) = 2 on S\E.

The following notation will be used throughout this paper. For a semigroup S, E = \{x: x \in S, x^2 = x\} and for e \in E, H(e) is the maximal subgroup of S containing the idempotent e. H = \bigcup \{H(e): e \in E\} and functions γ and θ are defined as in [5], that is, for x \in H, γ(x) is the idempotent of the unique maximal subgroup to which x belongs and θ(x) is the inverse of x in this group.

The following theorem will be proved:

**Theorem.** Let S be a compact semigroup with the properties:

1. S = H,
2. for e \in E, H(e) is totally disconnected,

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