A function $u = u(x)$, $x = (x_1, \ldots, x_n)$, is said to have bounded mean oscillation on a bounded cube $C_0$ if $u(x)$ is integrable over $C_0$ and there is a constant $K$ such that for every parallel subcube $C$, and some constant $a_C$, the inequality

$$\int_C |u(x) - a_C| \, dx \leq KR^n$$

holds, $R$ being the edge length of $C$. Such functions have been investigated by John and Nirenberg [1]. Their result states that if $u(x)$ has bounded mean oscillation on $C_0$ and satisfies (1) then the function

$$\mu(\sigma) = \text{meas}\{ |u(x) - a_{C_0}| > \sigma \}$$

("meas" means Lebesgue measure) satisfies

$$\mu(\sigma) \leq B R_0^{n-\epsilon} e^{-b\sigma/K},$$

where $R_0$ is the edge length of $C_0$ and $B$, $b$ are constants depending only on $n$.

In this paper I show that if $u(x)$ satisfies an inequality of the form (1) with $R^n$ replaced by $R^{n+\epsilon}$, $\epsilon > 0$, then $u$ is Hölder continuous with exponent $\epsilon$ (this condition is of course also necessary for Hölder continuity). Morrey's Lemma then follows as a simple corollary. The method of proof is essentially the same as that of John and Nirenberg and is based on the following decomposition lemma, a proof of which can be found in their paper.

**Lemma.** Let $u(x)$ be an integrable function on the bounded cube $C_0$ and let $s$ be a positive number such that

$$s \geq R_0^{-n} \int_{C_0} |u(x)| \, dx.$$

There then exists a denumerable set of open disjoint parallel subcubes $I_k$ ($k=1, 2, \cdots$) such that

(i) $|u(x)| \leq s$ a.e. in $C_0 - \bigcup_k I_k$,
(ii) the average, $u_k$, over $I_k$ satisfies $|u_k| \leq 2^n s$,

(iii) $\sum_k R_k^2 \leq s^{-1}\int_{C_k} |u(x)| \, dx$ ($R_k$ = edge length of $I_k$).

**Theorem.** Let $u = u(x)$ be an integrable function on a bounded cube $C_0$. Assume there exists a nondecreasing function $K(R)$ and a constant $\varepsilon$, $0 < \varepsilon \leq 1$, such that for every parallel subcube $C$ and some constant $a_C$ the inequality

$$\int_C |u(x) - a_C| \, dx \leq K(R) R^{n+\varepsilon}$$

holds, $R$ being the edge length of $C$. Then there is a function $v(x) = u(x)$ a.e. in $C_0$, such that

$$|v(x) - v(y)| \leq K_1 K(|x - y|) |x - y|^{\varepsilon}$$

holds for all points $x$, $y$ in $C_0$, with $K_1$ depending only on $\varepsilon$ and $n$.

The function $K(R)$ may tend to zero as $R \to 0$ in which case $v(x)$ is better than Hölder continuous.

**Proof.** If inequality (3) holds then it will also hold with $K(R)$ replaced by the constant $K(R_0)$. We call this constant $K$.

Since

$$\int_C |u(x)| \, dx - ac R^n \leq \int_C |u(x) - ac| \, dx,$$

it follows that $u_C$, the mean value of $u(x)$ over $C$, satisfies $|u_C - ac| \leq KR^{\varepsilon}$. Hence

$$\int_C |u(x) - u_C| \, dx \leq 2KR^{n+\varepsilon}.$$

Let $\Gamma = \Gamma(K; \varepsilon; R_0)$ be the class of all integrable functions $u(x)$ satisfying the condition (5) on some cube $C_0$ of edge length $R_0$. Let $\mu(\sigma) = \mu(\sigma; K; \varepsilon; R_0)$ be defined by

$$\mu(\sigma) = \sup_{u \in \Gamma(K; \varepsilon; R_0)} \text{meas}\{|u(x) - u_C| > \sigma\}.$$ 

Now multiply both sides of (5) by an arbitrary positive constant $K'$ and set $w(x) = K'u(x)$. It is clear that $w$ satisfies

$$\int_C |w(x) - w_C| \, dx \leq 2KK'R^{n+\varepsilon}$$

and therefore $\mu(\sigma; K; \varepsilon; R_0) = \mu(\sigma K'; KK'; \varepsilon; R_0)$. Substituting $\sigma/K'$ for $\sigma$ in this equation, we get
Next, perform a similarity transformation \( y = (R'/R_0)x \) which carries the cube \( C_0 \) onto a cube \( C' \) of edge length \( R' \), and set \( w(y) = u((R_0/R')y) \). \( w(y) \) satisfies

\[
\int_{C} |w(y) - w_C| \, dy \leq 2K \left( \frac{R_0}{R'} \right)^n R^{n+s}
\]

for every parallel subcube \( C \) of \( C' \) with edge length \( R \). It easily follows that

\[
\mu(\sigma; K(R_0/R')^s; \epsilon; R') = (R'/R_0)^n \mu(\sigma; K; \epsilon; R_0).
\]

Substituting \( K(R'/R_0)^s \) for \( K \) gives

\[
\mu(\sigma; K; \epsilon; R') = (\frac{R'}{R_0})^n \mu \left( \sigma; \frac{R'}{R_0} \right)^s ; \epsilon; R_0 \).
\]

Let \( \sigma \) and \( s \) be arbitrary numbers such that

\[
2^{-n} \sigma \geq s \geq R_0^{-n} \int_{C_0} |u(x)| \, dx.
\]

From the decomposition lemma we then have

\[
\text{meas}\{ |u(x)| > \sigma; x \in C_0\} \leq \sum_k \text{meas}\{ |u(x) - u_k| > \sigma - 2^ns; x \in I_k\}.
\]

If we assume, as we may, that \( u_{C_0} = 0 \), (9) then implies

\[
\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \mu(\sigma - 2^ns; K; \epsilon; R_k).
\]

From (8) we then have

\[
\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \left( \frac{R_k}{R_0} \right)^n \mu \left( \sigma - 2^ns; K \left( \frac{R_k}{R_0} \right)^s ; K; \epsilon; R_0 \right)
\]

and from (7) we further deduce

\[
\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \left( \frac{R_k}{R_0} \right)^n \mu \left( \sigma - 2^ns; \left( \frac{R_0}{R_k} \right)^s ; K; \epsilon; R_0 \right).
\]

Statement (iii) in the decomposition lemma gives

\[
\left( \frac{R_0}{R_k} \right)^s \geq s^{1/n} M, \quad M = R_0^s \left( \int_{C_0} |u(x)| \, dx \right)^{-1/n}.
\]
Using the fact that $\mu$ is nonincreasing in $\sigma$, we then have from (10), (11) and (iii) of the decomposition lemma

\begin{equation}
\mu(\sigma) \leq s^{-1}R_{0}^{-n} \int_{c_{0}} \left| u(x) \right| dx \cdot \mu((\sigma - 2n^{s})^{s}/M).
\end{equation}

Set $\sigma = 2^{n+1}s$. Then $(\sigma - 2n^{s})^{s}/M = s/(n \cdot M/2 \cdot \sigma)$. Thus, if we set

\begin{equation}
s = \left( \frac{2^{n^{s}}}{M} \right)^{s} = 2^{n^{s}}R_{0}^{-n} \int_{c_{0}} \left| u(x) \right| dx
\end{equation}

we get

\begin{equation}
\mu(\sigma) \leq 2^{-s/n} \mu(\sigma).
\end{equation}

Therefore $\mu = 0$ for $\sigma = 2^{n^{s}+1}R_{0}^{-n} \int_{c_{0}} \left| u(x) \right| dx$, or in other words

\begin{equation}
\left| u(x) - u_{c_{0}} \right| \leq 2^{n^{s}+n+3}K(R_{0})R_{1}^{s}
\end{equation}
a.e. in $C_{0}$. Therefore

\begin{equation}
\left| u(x) - u(y) \right| \leq 2^{n^{s}+n+3}K(R_{0})R_{1}^{s}
\end{equation}

for almost all $x$ and $y$ in $C_{0}$. Since $C_{0}$ is an arbitrary cube and since any two points $x$, $y$ with $|x - y| = R$ can be inclosed in a parallel subcube of edge length $R$ the desired result follows from (13).

**COROLLARY.** Let $u = u(x)$ have strong derivatives which are in $L^{p}$ $1 \leq p < \infty$ on a bounded cube $C_{0}$. Assume there is a nondecreasing function $K(K(R))$ and a constant $\epsilon$, $0 < \epsilon \leq 1$, such that for every parallel subcube $C$

\begin{equation}
\int_{C} \left| \nabla u(x) \right|^{p} dx \leq K(\epsilon)R^{n-p+\epsilon}
\end{equation}

holds, $R$ being the edge length of $C$. Then there is a function $v(x) = u(x)$ a.e. in $C_{0}$ such that

\begin{equation}
\left| v(x) - v(y) \right| \leq K_{2}K(\epsilon) \left| x - y \right|^{\epsilon}
\end{equation}

holds for all points $x, y$ in $C_{0}$ and $K_{2}$ depends only on $\epsilon$ and $n$.

**PROOF.** It is a simple matter to prove the Wirtinger inequality

\begin{equation}
\int_{C} \left| u(x) - u_{c_{0}} \right| dx \leq K_{3}R \int_{C} \left| \nabla u(x) \right| dx,
\end{equation}

with $K_{3}$ depending only on $n$. Applying the Hölder inequality to the right side of (16) we get
\[(17) \quad \int |u(x) - uc| \, dx \leq K_2 R^{n+1-n/\eta} \left( \int |\text{grad} \, u(x)|^p \, dx \right)^{1/p}\]

and the desired result follows from the previous theorem.

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**Reference**


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**PHRAGMÉN-LINDELOF THEOREMS FOR SECOND ORDER QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS**

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Phragmén-Lindelöf theorems for uniformly elliptic partial differential equations have been the subject of several papers in recent years (see e.g. [3; 4; 6; 7; 8; 10]). Here we are concerned with the Phragmén-Lindelöf theorem for second order quasi-linear elliptic equations of the form

\[(1) \quad L[u] = \sum a_{ij}(x, \rho)\partial_{x_i x_j} = f(x, z, \rho),\]

which need not be uniformly elliptic. The main result is Theorem 1 which roughly says that if \(u(x)\) is a subfunction with respect to (1) in a domain \(D\) contained in a half space and if \(u(x) \leq 0\) on the boundary of \(D\) then either \(u(x) \leq 0\) throughout \(D\) or the maximum of \(u(x)\) on a sphere of radius \(r\) is of order not less than \(r^\eta\) for some \(\eta > 0\). Probably the most interesting feature of this theorem is that its proof essentially depends only on the behavior of the functions \(a_{ij}(x, \rho)\) and \(f(x, z, \rho)\) for \(\sum \rho_i^2 \leq 1\). For \(f \equiv 0\) and dimension \(n = 2\) it is shown that \(\eta = 1\).

Let \(D\) be an unbounded domain contained in a half space of \(n\)-dimensional Euclidean space and let \(T\) be the domain in \(2n\)-dimen-

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