MEAN OSCILLATION OVER CUBES AND
HÖLDER CONTINUITY

NORMAN G. MEYERS

A function \( u = u(x), \quad x = (x_1, \ldots, x_n), \) is said to have bounded mean oscillation on a bounded cube \( C_0 \) if \( u(x) \) is integrable over \( C_0 \) and there is a constant \( K \) such that for every parallel subcube \( C \), and some constant \( a_c \), the inequality

\[
\int_C |u(x) - a_c| \, dx \leq KR^n
\]

holds, \( R \) being the edge length of \( C \). Such functions have been investigated by John and Nirenberg [1]. Their result states that if \( u(x) \) has bounded mean oscillation on \( C_0 \) and satisfies (1) then the function

\[
\mu(\sigma) = \text{meas}\{ |u(x) - a_{c_0}| > \sigma \}
\]

(“meas” means Lebesgue measure) satisfies

\[
\mu(\sigma) \leq B\sigma^{n-\epsilon}e^{-\epsilon R_0/K},
\]

where \( R_0 \) is the edge length of \( C_0 \) and \( B, b \) are constants depending only on \( n \).

In this paper I show that if \( u(x) \) satisfies an inequality of the form (1) with \( R^n \) replaced by \( R^{n+\epsilon}, \epsilon > 0 \), then \( u \) is Hölder continuous with exponent \( \epsilon \) (this condition is of course also necessary for Hölder continuity). Morrey’s Lemma then follows as a simple corollary. The method of proof is essentially the same as that of John and Nirenberg and is based on the following decomposition lemma, a proof of which can be found in their paper.

**Lemma.** Let \( u(x) \) be an integrable function on the bounded cube \( C_0 \) and let \( s \) be a positive number such that

\[
s \geq R_0^{-n} \int_{C_0} |u(x)| \, dx.
\]

There then exists a denumerable set of open disjoint parallel subcubes \( I_k (k = 1, 2, \ldots) \) such that

\[
(i) \quad |u(x)| \leq s \ a.e. \ in \ C_0 - \sum_k I_k.
\]

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Theorem. Let \( u = u(x) \) be an integrable function on a bounded cube \( C_0 \). Assume there exists a nondecreasing function \( K(R) \) and a constant \( \epsilon, 0 < \epsilon \leq 1 \), such that for every parallel subcube \( C \) and some constant \( a_c \) the inequality

\[
\int_C \left| u(x) - a_c \right| \, dx \leq K(R)R^{n+\epsilon}
\]

holds, \( R \) being the edge length of \( C \). Then there is a function \( v(x) = u(x) \) a.e. in \( C_0 \), such that

\[
\left| v(x) - v(y) \right| \leq K_1K(\left| x - y \right|) \left| x - y \right|^{\epsilon}
\]

holds for all points \( x, y \) in \( C_0 \), with \( K_1 \) depending only on \( \epsilon \) and \( n \).

The function \( K(R) \) may tend to zero as \( R \to 0 \) in which case \( v(x) \) is better than Hölder continuous.

Proof. If inequality (3) holds then it will also hold with \( K(R) \) replaced by the constant \( K(R_0) \). We call this constant \( K \).

Since

\[
\left| \int_C u(x)dx - a_cR^n \right| \leq \int_C \left| u(x) - a_c \right| \, dx,
\]

it follows that \( u_c \), the mean value of \( u(x) \) over \( C \), satisfies \( \left| u_c - a_c \right| \leq KR^\epsilon \). Hence

\[
\int_C \left| u(x) - u_c \right| \, dx \leq 2KR^{n+\epsilon}.
\]

Let \( \Gamma = \Gamma(K; \epsilon; R_0) \) be the class of all integrable functions \( u(x) \) satisfying the condition (5) on some cube \( C_0 \) of edge length \( R_0 \). Let \( \mu(\sigma) = \mu(\sigma; K; \epsilon; R_0) \) be defined by

\[
\mu(\sigma) = \sup_{u \in \Gamma(K; \epsilon; R_0)} \text{meas}\left\{ \left| u(x) - u_c \right| > \sigma \right\}.
\]

Now multiply both sides of (5) by an arbitrary positive constant \( K' \) and set \( w(x) = K'u(x) \). It is clear that \( w \) satisfies

\[
\int_C \left| w(x) - w_c \right| \, dx \leq 2KK'R^{n+\epsilon}
\]

and therefore \( \mu(\sigma; K; \epsilon; R_0) = \mu(\sigma K'; KK'; \epsilon; R_0) \). Substituting \( \sigma/K' \) for \( \sigma \) in this equation, we get
Next, perform a similarity transformation $y = (R'/R_0)x$ which carries the cube $C_0$ onto a cube $C'$ of edge length $R'$, and set $w(y) = w((R_0/R')y)$. $w(y)$ satisfies

$$\int_C |w(y) - w_c| \, dy \leq 2K\left(\frac{R_0}{R'}\right)^n R^{n+s}$$

for every parallel subcube $C$ of $C'$ with edge length $R$. It easily follows that $\mu(\sigma; K(R_0/R')^n; \epsilon; R') = (R'/R_0)^n \mu(\sigma; K; \epsilon; R_0)$. Substituting $K(R'/R_0)^n$ for $K$ gives

$$\mu(\sigma; K; \epsilon; R') = \left(\frac{R'}{R_0}\right)^n \mu\left(\sigma; \frac{K'}{K_0}; \epsilon; R_0\right).$$

Let $\sigma$ and $s$ be arbitrary numbers such that

$$2^{-n} \sigma \geq s \geq R_0^{-n} \int_{C_0} |u(x)| \, dx.$$

From the decomposition lemma we then have

$$\text{meas}\{ |u(x)| > \sigma; x \in C_0\} \leq \sum_k \text{meas}\{ |u(x) - u_k| > \sigma - 2^n s; x \in I_k\}.$$

If we assume, as we may, that $u_{C_0} = 0$, (9) then implies

$$\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \mu(\sigma - 2^n s; K; \epsilon; R_k).$$

From (8) we then have

$$\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \left(\frac{R_k}{R_0}\right)^n \mu\left(\sigma - 2^n s; K\left(\frac{R_k}{R_0}\right); \epsilon; R_0\right)$$

and from (7) we further deduce

$$\mu(\sigma; K; \epsilon; R_0) \leq \sum_k \left(\frac{R_k}{R_0}\right)^n \mu\left(\sigma - 2^n s; \frac{R_0}{R_k}; K; \epsilon; R_0\right).$$

Statement (iii) in the decomposition lemma gives

$$\left(\frac{R_0}{R_k}\right)^n \leq s^n M, \quad M = R_0^s \left(\int_{C_0} |u(x)| \, dx\right)^{-s/n}. $$
Using the fact that \( \mu \) is nonincreasing in \( \sigma \), we then have from (10), (11) and (iii) of the decomposition lemma

\[
\mu(\sigma) \leq s^{-1}R_0^{-n} \int_{c_0} |u(x)| \, dx \cdot \mu((\sigma - 2^n)s^{s/n}M).
\]

Set \( \sigma = 2^{n+1}s \). Then \( (\sigma - 2^n)s^{s/n}M = s^{s/n} \cdot M/2 \cdot \sigma \). Thus, if we set

\[
s = \left( \frac{2}{M} \right)^{n/s} = 2^{n/s} \cdot R_0^{-n} \int_{c_0} |u(x)| \, dx
\]

we get

\[
\mu(\sigma) \leq 2^{-n/s} \mu(\sigma).
\]

Therefore \( \mu = 0 \) for \( \sigma = 2^{n/s+n+1}R_0^{-n} \int_{c_0} |u(x)| \, dx \), or in other words

\[
|u(x) - u_{c_0}| \leq 2^{n/n+n+2} K(R_0) R_0^a
\]

a.e. in \( C_0 \). Therefore

\[
(13) \quad |u(x) - u(y)| \leq 2^{n/\epsilon+n+3} K(R_0) R_0^a
\]

for almost all \( x \) and \( y \) in \( C_0 \). Since \( C_0 \) is an arbitrary cube and since any two points \( x, y \) with \( |x - y| = R \) can be inclosed in a parallel subcube of edge length \( R \) the desired result follows from (13).

**Corollary.** Let \( u = u(x) \) have strong derivatives which are in \( L^p \) \((1 \leq p < \infty)\) on a bounded cube \( C_0 \). Assume there is a nondecreasing function \( K = K(R) \) and a constant \( \epsilon \), \( 0 < \epsilon \leq 1 \), such that for every parallel subcube of \( C_0 \)

\[
(14) \quad \int_C |\text{grad } u(x)|^p \, dx \leq K^p(R) R^{(n-p)+p \epsilon}
\]

holds, \( R \) being the edge length of \( C \). Then there is a function \( v(x) = u(x) \) a.e. in \( C_0 \) such that

\[
(15) \quad |v(x) - v(y)| \leq K_2 K(|x - y|) |x - y|^\epsilon
\]

holds for all points \( x, y \) in \( C_0 \) and \( K_2 \) depends only on \( \epsilon \) and \( n \).

**Proof.** It is a simple matter to prove the Wirtinger inequality

\[
(16) \quad \int_C |u(x) - u_c| \, dx \leq K_3 R \int_C |\text{grad } u(x)| \, dx
\]

with \( K_3 \) depending only on \( n \). Applying the Hölder inequality to the right side of (16) we get.
(17) \[ \int_{\partial} |u(x) - u_0| \, dx \leq K \varepsilon R^{n+1-n/2} \left( \int_{\partial} |\nabla u(x)|^p \, dx \right)^{1/p} \]

and the desired result follows from the previous theorem.

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**Reference**


**University of Minnesota**

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**Phragmén-Lindelöf Theorems for Second Order Quasi-Linear Elliptic Partial Differential Equations**

**John O. Herzog**

Phragmén-Lindelöf theorems for uniformly elliptic partial differential equations have been the subject of several papers in recent years (see e.g. [3; 4; 6; 7; 8; 10]). Here we are concerned with the Phragmén-Lindelöf theorem for second order quasi-linear elliptic equations of the form

\[ L[u] = \sum a_{ij}(x, \partial u\partial x) = f(x, u, \partial u\partial x), \]

which need not be uniformly elliptic. The main result is Theorem 1 which roughly says that if \( u(x) \) is a subfunction with respect to (1) in a domain \( D \) contained in a half space and if \( u(x) \leq 0 \) on the boundary of \( D \) then either \( u(x) \leq 0 \) throughout \( D \) or the maximum of \( u(x) \) on a sphere of radius \( r \) is of order not less than \( r^\eta \) for some \( \eta > 0 \).

Probably the most interesting feature of this theorem is that its proof essentially depends only on the behavior of the functions \( a_{ij}(x, \partial u\partial x) \) and \( f(x, u, \partial u\partial x) \) for \( \sum \partial_r \leq 1 \). For \( f \equiv 0 \) and dimension \( n = 2 \) it is shown that \( \eta = 1 \).

Let \( D \) be an unbounded domain contained in a half space of \( n \)-dimensional Euclidean space and let \( T \) be the domain in \( 2n \)-dimen-

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