THE k-NORM EXTENSION PROPERTY FOR BANACH SPACES[1]

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1. Introduction. The letter k will denote an infinite cardinal number. By a *k-family* (*k-set*) is meant a family (set) whose cardinal number does not exceed k. If B is a set and $\mathcal{C}$ a family of subsets of B, we say that $\mathcal{C}$ has the *binary intersection property* (*k-binary intersection property*) iff any nonvoid subfamily (k-subfamily) of $\mathcal{C}$ whose members meet pairwise has a nonvoid intersection.

Let B be a normed linear space (we consider only real normed linear spaces). For each b in B and each non-negative real number r, $S(b, r)$ denotes $\{x \in B: \|x-b\| \leqq r\}$, the *closed sphere* of B centered at b of radius r. We say that B is *k-separable* iff B has a dense k-subset. And B has the *norm extension property* (*k-norm extension property*) iff any continuous linear operator into B from a subspace of a (k-separable) normed linear space Y can be extended to a linear operator of the same norm on all of Y. Nachbin proved the following theorem in [3].

THEOREM 1. *The following statements are equivalent for a normed linear space B.*

(i) *B has the norm extension property.*

(ii) *If Y is a normed linear space of the form $B \oplus [x]$, there is a linear operator H: $Y \to B$ of norm* 1 *such that $H(b) = b$ for all b in B.*

(iii) *The family of closed spheres of B has the binary intersection property.*

We remark that in [1],[2] Aronszajn states that he proved the equivalence of the conditions (i) and (iii) above in his doctoral dissertation (1929) but that the result was never published. In this note we prove

THEOREM 2. *The following statements are equivalent for a normed linear space B.*

(i) *B has the k-norm extension property.*

(ii) *If M is a k-separable subspace of B and if Y is a normed linear*

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space of the form $M \oplus [x]$, *there is a linear operator* $H: Y \to B$ *of norm* 1 *such that* $H(m) = m$ *for all* m *in* M.

(iii) *The family of closed spheres of* B *has the* k*-binary intersection property.*

We then consider the Banach space $C^*(S)$ of all bounded continuous real-valued functions on a topological space S and show that $C^*(S)$ has the k-norm extension property iff it has the k-extension property as a partially ordered set (§3). When S is compact and Hausdorff, a slight modification of a theorem in [1] yields a characterization of those S such that $C^*(S)$ has the k-norm extension property. Finally, if S is compact, Hausdorff, and if the set $\mathfrak{Q}(S)$ of all closed and open subsets of S forms a base for the open sets of S, then $C^*(S)$ has the k-norm extension property iff $\mathfrak{Q}(S)$ has the k-extension property as a Boolean Algebra (§4).

2. **Proof of Theorem 2.** It is obvious that (i) implies (ii). To prove that (ii) implies (iii), assume that B satisfies (ii) but that there is a nonvoid k-family $\mathfrak{C}$ of closed spheres of B whose members meet pairwise but whose intersection is void. Let C denote the set of centers of the spheres of $\mathfrak{C}$ and let M denote the subspace of B spanned by C. Under these circumstances, Nachbin constructs (in [3]) a normed linear space $Y = M \oplus [x]$ and a real-valued function p on M satisfying (1) $p(m) = \|m - x\|$ for all m in M and (2) for each b in B, there is an m in M such that $p(m) < \|m - b\|$. But Y has a dense k-subset (e.g., the set of linear combinations of $C \cup \{x\}$ with rational coefficients), so the operator $i(m) = m$ carrying $M \subset Y$ onto $M \subset B$ extends to a continuous linear operator $I: Y \to B$ of norm 1. Hence, $p(m) = \|m - x\| \geqq \|I(m - x)\| = \|m - I(x)\|$ for all m in M contradicting (2) above (set $b = I(x)$).

To prove that (iii) implies (i), assume (iii). Let Y be a k-separable normed linear space of the form $M \oplus [x]$, and let $h: M \to B$ be a linear operator of norm 1. We will extend h to a linear operator $H: Y \to B$ of norm 1; the condition (i) is easily proven from this by normalizing operators and using in a familiar Zorn's Lemma argument the fact that a subspace of a k-separable normed linear space is k-separable. Proceeding, let C be a dense k-subset of M and for each u in $h[C]$ define $p(u) = \inf\{\|c + x\| : c \in C, h(c) = u\}$. We assert that the members of the family $\{S(u, p(u)) : u \in h[C]\}$ meet pairwise. For let u and v belong to $h[C]$. Then for any c and d in C such that $h(c) = u$ and $h(d) = v$, $\|u - v\| = \|h(c) - h(d)\| \leqq \|c - d\| \leqq \|c + x\| + \|d + x\|$; hence, $\|u - v\| \leqq p(u) + p(v)$. Since the cardinal number of $h[C]$ does not exceed k, there is an element $-b$ in $\bigcap S(u, p(u))$ $(u \in h[C])$; i.e., $\|h(c) + b\| \leqq p(h(c))$ for every c in C. Consequently, by the definition

of p, $\|h(c)+b\| \leqq \|c+x\|$ for every c in C. The denseness of C in M and the continuity of the real-valued function $m \to \|h(m)+b\| - \|m+x\|$ implies $\|h(m)+b\| \leqq \|m+x\|$ for all m in M. Define $H: Y \to B$ by setting $H(m+rx) = h(m)+rb$. Then H is clearly a linear extension of h so that $\|H\| \geqq \|h\| = 1$. But for each $r \neq 0$ and m in M, $\|H(m+rx)\| = \|h(m)+rb\| = |r| \, \|h(m/r)+b\| \leqq |r| \, \|m/r+x\| = \|m+rx\|$; hence, $\|H\| \leqq 1$. Therefore, $\|H\| = 1$.

COROLLARY 1. *If a normed linear space B has the k-norm extension property, it is complete.*

PROOF. The idea of this proof is due to Aronszajn and Panitchpakdi. Let x_i be a Cauchy sequence in B. For each i, define $r(x_i) = \sup \|x_i - x_j\|$ $(j > i)$. If $p < q$, $r(x_p) + r(x_q) \geqq \|x_p - x_q\|$; hence, the members of the countable family $\{S(x_i, r(x_i))\}$ of closed spheres of B meet pairwise. By hypothesis and Theorem 2, there is an element b in $\bigcap S(x_i, r(x_i))$; i.e., $\|x_i - b\| \leqq r(x_i)$ for all i. To prove that the sequence x_i converges to b, let $\epsilon > 0$ be given. Using the fact that x_i is Cauchy, choose p so large that $2r(x_p) < \epsilon$. Then for any integer n larger than p, $\|x_n - b\| \leqq \|x_n - x_p\| + \|b - x_p\| \leqq 2r(x_p) < \epsilon$.

COROLLARY 2. *A k-separable Banach space with the k-norm extension property has the norm extension property.*

PROOF. If B is such a space, it satisfies the condition (ii) of Theorem 2. Substituting $M = B$, as we may, in the statement of this condition, we obtain condition (ii) of Theorem 1.

COROLLARY 3. *If a separable Banach space B has the k-norm extension property, it is finite dimensional.*

PROOF. If B is such a space, then B is k_0-separable and has the k_0-norm extension property, where k_0 denotes the first infinite cardinal. By Corollary 2, B has the norm extension property. By a result [2] of D. B. Goodner, B is finite dimensional.

3. **The k-extension property for partially ordered sets.** A *partially* ordered set, of course, is a set B together with a reflexive, antisymmetric, and transitive relation "$\leqq$" on B. A function h from one partially ordered set A into another is called *order preserving* iff x and y in A and $x \leqq y$ imply $h(x) \leqq h(y)$.

Let B be a partially ordered set. If S and T are subsets of B, $S \leqq T$ signifies that $s \leqq t$ for all s in S and t in T. We say that B has the *interpolatory property* iff s_1, s_2, t_1, t_2 in B with $\{s_1, s_2\} \leqq \{t_1, t_2\}$ imply the existence of an element b in B such that $\{s_1, s_2\} \leqq \{b\} \leqq \{t_1, t_2\}$. If x and y are in B with $x \leqq y$, $[x, y]$ denotes $\{b \in B: x \leqq b \leqq y\}$, the

closed interval of B determined by x and y. We say that B has the *k-extension property* iff any order preserving function into B from a subset of a partially ordered k-set Y can be extended to an order preserving function on all of Y.

THEOREM 3. *For a partially ordered set B, the first two statements below are equivalent and they imply the third. If B has the interpolatory property, all three statements are equivalent.*

(i) *B has the k-extension property.*

(ii) *If S and T are k-subsets of B with $S \leqq T$, there is an element b in B such that $S \leqq \{b\} \leqq T$.*

(iii) *The family of closed intervals of B has the k-binary intersection property.*

PROOF. (i) implies (ii): Let S and T be k-subsets of B with $S \leqq T$. Let w be an element not in B and extend the ordering of $S \cup T$ to a partial ordering of $Y = S \cup \{w\} \cup T$ by decreeing $S \leqq \{w\} \leqq T$. Y is a k-set so by hypothesis, the inclusion map $S \cup T \subset B$ extends to an order preserving function $h: Y \to B$. Consequently, $S = h[S] \leqq h(w) \leqq h[T] = T$, and $h(w)$ is the required element of B.

(ii) implies (i): Let Y be a partially ordered k-set of the form $M \cup \{x\}$ and let $h: M \to B$ be an order preserving function. Set $S = \{m \in M: m \leqq x\}$ and $T = \{m \in M: m \geqq x\}$. Then $h[S] \leqq h[T]$, and since $h[S]$ and $h[T]$ are k-sets, there is an element b in B such that $h[S] \leqq \{b\} \leqq h[T]$. Extend h to an order preserving function $H: Y \to B$ by setting $H(x) = b$. The general statement of (i) now follows from a Zorn's Lemma argument.

(ii) implies (iii): Let $\{[s_i, t_i]: i \in I\}$ be a nonvoid k-family of closed intervals of B which meet pairwise. Then $\{s_i: i \in I\} \leqq \{t_i: i \in I\}$ since if i and j are in I, there is an element in $[s_i, t_i] \cap [s_j, t_j]$. By hypothesis, there is an element b in B such that $\{s_i: i \in I\} \leqq \{b\} \leqq \{t_i: i \in I\}$; hence, b is in $\bigcap [s_i, t_i]$ $(i \in I)$.

Now assume B has the interpolatory property. (iii) implies (ii): Let S and T be k-subsets of B with $S \leqq T$. The members of the k-family $\{[s, t]: s \in S, t \in T\}$ meet pairwise by the interpolatory property; hence, there is an element b in $\bigcap [s, t]$ $(s \in S, t \in T)$. Thus, $S \leqq \{b\} \leqq T$.

THEOREM 4. *$C^*(S)$ has the k-norm extension property if and only if it has the k-extension property as a partially ordered set.*

PROOF. The partial ordering of $C^*(S)$ is defined by $f \leqq g$ iff $f(s) \leqq g(s)$ for all s in S; indeed, $C^*(S)$ is a lattice and therefore has the interpolatory property. And $C^*(S)$ is a Banach space in the usual sup norm, $\|f\| = \sup |f(s)|$ $(s \in S)$. For each real number r, $\boldsymbol{r}$ will denote the function on S constant at r.

Assume first that $C^*(S)$ has the k-extension property as a partially ordered set. Let $\mathfrak{Q}$ be a nonvoid k-family of closed spheres of $C^*(S)$ whose members meet pairwise. Then since every closed sphere of $C^*(S)$ is a closed interval, $\bigcap\mathfrak{Q}$ is nonvoid by hypothesis and Theorem 3. By Theorem 2, $C^*(S)$ has the k-norm extension property.

Conversely, assume $C^*(S)$ has the k-norm extension property. Let $\mathfrak{F}$ and $\mathfrak{G}$ be k-subsets of $C^*(S)$ such that $\mathfrak{F}\leqq\mathfrak{G}$. To apply Theorem 3, an element of $C^*(S)$ must be found in between $\mathfrak{F}$ and $\mathfrak{G}$. For each f in $\mathfrak{F}$ and g in $\mathfrak{G}$, set $[f, g, f+]$ equal to $[f, f+r]$ for some positive r so large that the interval contains g. And set $[g-, f, g]$ equal to $[g-r, g]$ for some positive r so large that the interval contains f. Each of *these* closed intervals is a closed sphere of $C^*(S)$. Furthermore, using the fact that $C^*(S)$ is a lattice, one proves that the members of this k-family $\mathfrak{Q}$ of closed spheres meet pairwise. For example, $[f_1, g_1, f_1+]\cap[f_2, g_2, f_2+]$ contains $f_1\vee f_2$ and $[f_1, g_1, f_1+]\cap[g_2-, f_2, g_2]$ contains $g_1\wedge g_2$. By Theorem 2, there is a function h in $\bigcap\mathfrak{Q}$. Given f in $\mathfrak{F}$ and g in $\mathfrak{G}$, h is a member of $[f, g, f+]\cap[g-, f, g]$; hence, $f\leqq h\leqq g$. Thus $\mathfrak{F}\leqq\{h\}\leqq\mathfrak{G}$.

4. **The k-extension property for Boolean Algebras.** Our reference on the subject of Boolean Algebras and homomorphisms is [**4**]. A Boolean Algebra B is said to have the *k-extension property* iff any homomorphism into B from a subalgebra of a Boolean Algebra Y, Y a k-set, can be extended to a homomorphism on all of Y.

THEOREM 5. *A Boolean Algebra B has the k-extension property if and only if the family of closed intervals of B has the k-binary intersection property.*

PROOF. If B satisfies the latter condition, then B satisfies condition (ii) of Theorem 3. Sikorski's proof of 33.1 on page 113 of [**4**] can be taken almost intact to prove that B has the k-extension property as a Boolean Algebra.

Let $\mathfrak{X}$ be a subset of B and let $B(\mathfrak{X})$ denote the subalgebra of B generated by $\mathfrak{X}$. From the characterization of the elements of $B(\mathfrak{X})$ given in Chapter I of [**4**], it follows that $B(\mathfrak{X})$ is a k-set when $\mathfrak{X}$ is.

Now assume B has the k-extension property as a Boolean Algebra and let S and T be k-subsets of B with $S\leqq T$. Let j be a 1-1 homomorphism carrying B into a complete Boolean Algebra E (see [**4**, p. 118]). Let e be an element of E such that $j[S]\leqq\{e\}\leqq j[T]$; for instance, $\sup j[S]$ (a homomorphism preserves order). The subalgebra $E_0\equiv E(j[S]\cup\{e\}\cup j[T])$ is a k-set so that j^{-1}, which carries $E(j[S]\cup j[T])$ into B, extends by hypothesis to a homomorphism $h: E_0\to B$. Given s in S and t in T, $s=j^{-1}\circ j(s)=h\circ j(s)\leqq h(e)$

$\leqq h \circ j(t) = j^{-1} \circ j(t) = t$; hence, $S \leqq h(e) \leqq T$. By Theorem 3, the family of closed intervals of B has the k-binary intersection property.

THEOREM 6 (ARONSZAJN AND PANITCHPAKDI). *Let S be compact and Hausdorff. The family of closed spheres of $C^*(S)$ has the k-binary intersection property if and only if* $\mathrm{cl}(\bigcup_i U_i) \cap \mathrm{cl}(\bigcup_j V_j)$ *is void for any two k-families* $\{U_i : i \in I\}$ *and* $\{V_j : j \in J\}$ *of open subsets of S such that* $\mathrm{cl}(U_r) \subset \bigcup_i U_i$ *for each i in I,* $\mathrm{cl}(V_t) \subset \bigcup_j V_j$ *for all j in J, and* $(\bigcup_i U_i) \cap (\bigcup_j V_j)$ *is void.*

PROOF. In the proof of Theorem 2, §5, [1], replace strict inequalities between cardinal numbers by inequalities.

COROLLARY. *Let S be compact and Hausdorff. Then $C^*(S)$ has the k-norm extension property if and only if S has the topological property of Theorem* 6.

THEOREM 7. *Suppose S is compact, Hausdorff, and the set $\mathfrak{Q}(S)$ of closed and open subsets of S form a base for the open sets of S. Then $C^*(S)$ has the k-norm extension property if and only if $\mathfrak{Q}(S)$ has the k-extension property as a Boolean Algebra.*

PROOF. It need only be shown that under our hypotheses on S, the topological condition on S of Theorem 6 is equivalent to the condition (ii) of Theorem 3 on the partially ordered set $\mathfrak{Q}(S)$. This equivalence is easily proven from the fact that if N is a closed subset of the open subset U of S, there is a closed and open subset L of S such that $N \subset L \subset U$.

REMARK. If S represents a k-complete Boolean Algebra, then $\mathfrak{Q}(S)$ has the k-extension property; hence, $C^*(S)$ has the k-norm extension property. Since there is a good supply of k-complete Boolean Algebras, there is a good supply of Banach spaces with the k-norm extension property.

PROBLEM. Can every Banach space with the k-norm extension property be represented as a $C^*(S)$?

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