A NEW TYPE OF VECTOR FIELD AND
INVARIANT DIFFERENTIAL SYSTEMS

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In [1] Robert Hermann introduced the concept of tangent vector
fields on the space of maps of one manifold into another. A special
type of these are the "k-vector fields" which were studied in [3],
where this author defined their bracket and exponential. This paper
explores further the analogy with classical continuous groups. Specif-
ically, we study invariance of systems of partial differential equations
under k-vector fields.

1. Introduction. Every map and manifold is $C^\infty$ unless otherwise
noted. $J^k = J^k(N, M)$ is the manifold of k-jets $j^k_0(f)$ of order k of maps
$f: N \to M$ from the manifold $N$ to the manifold $M$. $\alpha$ and $\beta$ are the
source and target projections, $\rho^{k+i}: J^{k+i} \to J^i$ the usual projection.
$T(M)$ denotes the tangent bundle to $M$, $M_y$ the tangent space at
$y \in M$, $\pi$ the tangent bundle projection. $C^\infty(Q)$ is the algebra (over
the reals $R$) of $C^\infty$ real-valued functions on the manifold $Q$.

A k-vector field is a map $\theta: C^\infty(M) \to C^\infty(J^k)$ which is linear over $R$
and satisfies

$$
\theta(FG) = (F \circ \beta)(G) + (G \circ \theta)(F).
$$

In [3] the $i$th prolongation $P^i\theta: C^\infty(J^i) \to C^\infty(J^{i+k})$ was defined. This
satisfies, for $H \in C^\infty(M)$, $F$ and $G \in C^\infty(J^i)$, $P^i\theta(FG) = (F \circ \rho^{k+i})P^i\theta(G) + (G \circ \rho^{k+i})P^i\theta(F)$ and
$P^i\theta(H \circ \beta) = \theta(H) \circ \rho^{k+i}$. Using these facts one
sees that if $\theta$ and $\psi$ are $k$- and $i$-vector fields, respectively, then
$[\theta, \psi] = P^i\theta \circ \psi - P^i\psi \circ \theta$ is a $(k+i)$-vector field.

In local coordinates $(x^i)$ on $N$, $(y^\lambda)$ on $M$, $(x^i, y^\lambda, \rho^1, \ldots, \rho^s)$
on $J^k$, where $i, j_1, \ldots, j_k = 1, \ldots, n$; $\lambda = 1, \ldots, m$, we follow
Kuranishi [4] in defining for each $F \in C^\infty(J^k)$, $\partial^l_i F \in C^\infty(J^k+1)$ by

$$
\partial^l_i F = \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial y^\lambda} \rho^{\lambda}_{j_1} + \cdots + \frac{\partial F}{\partial \rho^{j_1 \cdots j_{k+1}}} \rho^{j_1 \cdots j_{k+1}}.
$$

If $\theta$ is a k-vector field, then in local coordinates $\theta$ may be expressed
in the form $\theta = a^\lambda(\partial/\partial y^\lambda)$ where the $a^\lambda$ are real-valued functions on the
coordinate neighborhood $U$ in $J^k$. If $F \in C^\infty(M)$, then $\theta(F)$ on $U$ is

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the function \(a^\lambda \partial F/\partial y^\lambda \circ \beta\). In these local coordinates the \(i\)th prolongation of \(\theta\) has the expression
\[
P^i \theta = a^\lambda \frac{\partial}{\partial y^\lambda} + \partial_j \cdots \partial_{j_i} \frac{\partial}{\partial p_{j_i}} + \cdots + \partial_{j_i} \cdots \partial_{j_i} \frac{\partial}{\partial p_{j_i}}
\]
(See [3, Lemma 1].) We shall also need the following lemma whose proof we omit.

**Lemma 1.** Let \(\theta\) be a \(k\)-vector field, \(F_1, F_2 \in C^\infty(J^i), G \in C^\infty(M)\) and \(F \in C^\infty(J^{i-j})\), where \(0 < j < i\). Then
(A) \(P^i \theta(F \circ \rho^i_{-j}) = (P^i \theta(F)) \circ \rho^i_{i-j+k}\),
(B) \(P^i \theta(G \circ \beta) = \theta(G) \circ \rho^i_{i+k}\),
(C) \(P^i \theta(F_1 F_2) = (F_1 \circ \rho^i_{i+k}) P^i \theta(F_2) + (F_2 \circ \rho^i_{i+k}) P^i \theta(F_1),\)
(D) \(P^i \theta(\partial_{j_1}^{\lambda_1} \cdots \partial_{j_i}^{\lambda_i} G \circ \rho^i_{-j}) = \partial_{j_1}^{\lambda_1} \cdots \partial_{j_i}^{\lambda_i} \theta(G) \circ \rho^i_{i-j}, r < k,\)
(E) \(P^i \theta(\partial_{j_1}^{\lambda_1} \cdots \partial_{j_1}^{\lambda_1} F \circ \rho^i_{-j+k}) = (\partial_{j_1}^{\lambda_1} \cdots \partial_{j_1}^{\lambda_1} P^i \theta(F)) \circ \rho^i_{i-j+k+r}, r < j.\)
Conversely, if \(\phi: C^\infty(J^i) \to C^\infty(J^{i+k})\) satisfies (A), \(\cdots\), (E) when \(P^i \theta\) is replaced by \(\phi\), then \(\phi = P^i \theta\).

Another important property for us is that if \(F \in C^\infty(J^i), f: N \to M,\) then \((\partial/\partial x^i)G(j^i(f)) = (\partial_i G)(j^{i+1}(f))\) for all \(G \in C^\infty(J^i)\) [2, Proposition 1.10].

Let \(I = (-\epsilon, \epsilon)\). An **integral curve of \(\theta\)** starting at \(f_0: N \to M\) is a 1-parameter family \(f: N \times I \to M\) with \(f_0(x) = f(x, 0)\) and for every \(F \in C^\infty(M),\)
\[
\theta(f_0(x))(F) = \theta(F)(j^1(I)) = \frac{\partial}{\partial t}(F \circ f)(x, t) = \left(\left(f \ast \frac{\partial}{\partial t}\right) F\right)(x, t).
\]

2. **Differential systems.** A system \(\Sigma\) of partial differential equations (s.p.d.e.) of order \(h\) with \(N\) as independent and \(M\) as dependent variables is a finitely generated ideal in \(C^\infty(J^h)\). A solution of \(\Sigma\) is a map \(f: N \to M\) such that \(F(j^h(f)) = 0\) for all \(x \in N, F \in \Sigma\). \(P^i \Sigma\) denotes the s.p.d.e. of order \(h + k\) generated by the functions \(F \circ \rho^h_{i+k}, \partial_{j}^i F \circ \rho^h_{i+k}, \cdots, \partial_{j_i}^i F, 1 \leq j, j_i \leq n, F \in \Sigma.\)

**Definition.** A \(k\)-vector field \(\theta\) leaves \(\Sigma\) **invariant** if for each \(F \in \Sigma, P^i \theta(F) \in P^i \Sigma.\)

Compare with [2] for the older theory. The intuitive meaning of invariance under a transformation group was that the transformations permute the solutions. We shall show that if \(f_0\) is a solution of \(\Sigma\) which belongs to an integral curve of \(\theta\), then \(\Sigma\) evaluated at this integral curve has zero derivatives at \(f_0\) of all orders.

**Lemma 2.** If \(\theta\) is an invariant vector field of \(\Sigma\), then \(\theta\) is an invariant vector field for \(P^i \Sigma, i\).
This follows from (D) and (E) in Lemma 1. Using local coordinates, a calculation proves

**Lemma 3.** If $F \in C^\omega(J)$, $f: N \times I \to M$, and $(\partial f/\partial t) = \theta(j^k_x(f))$, then

$$\frac{\partial}{\partial t} F(j^k_x(f)) = P^i \theta(F) \big|_{j^i(x,t)}.$$

**Lemma 4.** If $f: N \to M$ is a solution of $\Sigma$, it is a solution of $P^* \Sigma$, all $i$.

**Theorem 1.** Suppose that

(A) $\theta$ is an invariant $k$-vector field of $\Sigma$,
(B) $f: N \times I \to M$ satisfies $(\partial f/\partial t) = \theta(j^k_x(f))$, and
(C) $f(1, 0): N \to M$ is a solution of $\Sigma$.

Then $(\partial^n/\partial t^n) F(j^k_x(f)) \big|_{t=0} = 0$ for all $x \in N$, $F \in \Sigma$, and $n = 1, 2, \ldots$.

**Proof.** From Lemma 3,

$$\frac{\partial}{\partial t} F(j^k_x(f)) = P^i \theta(F) \big|_{j^i(x,t)}.$$

However, $P^k \theta(F) \in P^* \Sigma$, and $f$ is a solution of $P^* \Sigma$ by Lemma 4. Hence $P^k \theta(F)(j^{k+h}_x(f)) \big|_{t=0} = 0$, all $x \in N$. Let $F^1 = P^k \theta(F) \in P^* \Sigma$. By Lemma 3,

$$P^{i+k} \theta(F) \big|_{j^i(x,t)} = \frac{\partial}{\partial t} F^1(j^{i+k}_x(f)) = \frac{\partial}{\partial t} \left[ F^1(j^k_x(f)) \right].$$

Using Lemma 4 as before, $(\partial^2/\partial t^2) F(j^k_x(f)) \big|_{t=0} = 0$. Continuing in this way, the result follows. Q.E.D.

When the manifolds and functions are real-analytic, Theorem 1 implies that integral curves of an invariant vector field which pass through one solution yield solutions for all parameter values.

3. Lie algebra structure.

**Proposition.** Let $\theta$ and $\psi$ be $k$- and $h$-vector fields, respectively. Then

$$P^i[\theta, \psi] = P^{i+k} \theta \circ P^i \psi - P^{i+h} \psi \circ P^i \theta.$$

**Proof.** By induction on $i$. A local coordinate calculation shows the result for $i = 1$. Call $\phi: C^\omega(J) \to C^\omega(J^{i+h+k})$ the operator on the right-hand side. We shall use Lemma 1. Let $F, F_1 \in C^\omega(J)$, $G \in C^\omega(M)$, and $F \in C^\omega(J^{i-1})$.

$$P^{i+k} \theta \circ P^i \psi(G \circ \rho_{i-1}) = P^{i+k} \theta(P^{i-k} \psi(G) \circ \rho_{i+k-i})$$

$$= (P^{i+h-k} \theta(P^{i-k} \psi(G))) \circ \rho_{i+h+k-i}$$

$$= (P^{i+h-k} \theta P^{i-k} \psi)(F) \circ \rho_{i+h+k-i},$$
applying Lemma 1(A) to $\psi$ and $\theta$. Interchanging $\theta$ and $\psi$, we find

$$P^i \phi(F \circ \rho_{i-j}) = (P^{i-j} \phi(F)) \circ \rho_{i+h+k-i}.$$  

Now, by induction, $P^{i-j} \phi(F) = P^{i-j} [\theta, \psi]$. Hence (A) holds for $\phi$. The same technique works for (B), $\cdots$, (E). Q.E.D.

**Theorem 2.** If $\theta$ and $\psi$ are $k$- and $h$-vector fields, respectively, which leave $\Sigma$ invariant, then $[\theta, \psi]$ leaves $\Sigma$ invariant.

**Proof.** If $F \in \Sigma$ and $\Sigma$ is of order $i$, then $P^i [\theta, \psi](F) = P^{i+h} \circ P^h \psi(F) - P^{i+h} \psi \circ P^h \theta(F)$. However, $P^h \psi(F) \in P^h \Sigma$. By Lemma 2, $\theta$ is an invariant vector field of $P^h \Sigma$, so $P^{i+h} \circ P^h \psi(F) \in P^{i+h} \Sigma$. Similarly $P^{i+h} \psi \circ P^h \theta(F) \in P^{i+h} \Sigma$. Q.E.D.

We conclude that the set of all $k$-vector fields, $k=1, 2, \cdots$, leaving $\Sigma$ invariant forms a Lie algebra under the bracket.

4. An example. Let $N = E^n$, $M = E^m$. Consider an s.p.d.e. of the type

$$\frac{\partial y^\mu}{\partial x^n} = \phi^i(x^1, \cdots, x^{n-1}, y^\mu, \frac{\partial y^\mu}{\partial x^1}, \cdots, \frac{\partial y^\mu}{\partial x^{n-1}}),$$

$\lambda, \mu = 1, \cdots, m$. On $J^1$ let $F^\lambda = \phi^i - \phi^i(x^1, y^\mu, \rho^\nu_r)$, and let $\Sigma$ be generated by $F^1, \cdots, F^m$. Then by a calculation one may check that $\theta = \phi^i(\partial / \partial y^i)$ turns out to be an invariant vector field of $\Sigma$.

We can see that $\theta$ generates solutions of the Cauchy problem associated with $\Sigma$. Since $\theta$ is independent of $x^n$ and $\rho^i_r$, it can be considered a 1-vector field on $E^{n-1}$. Suppose $f_0 : E^{n-1} \to E^m$ is the initial data at $x^n = 0$. Suppose $I = \{x^*| -\epsilon < x^n < \epsilon\}$ and $f : E^{n-1} \times I \to E^m$ is an integral curve of $\theta$ through $f_0$. But that is merely another way of saying that $f$ is a solution of $\Sigma$.

**References**


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