NOTE ON POINTWISE PERIODIC SEMIGROUPS

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An element $x$ in a semigroup $S$ is said to be periodic if there exists
a positive integer $n$ such that $x^{n+1} = x$, and the least such $n$, $p(x)$, is
the period of $x$. $S$ is pointwise periodic if each $x$ in $S$ is periodic. In
[4], A. D. Wallace asks the following question concerning pointwise
periodic topological semigroups.

Problem 3: If $S$ is a pointwise periodic semigroup and is topo-
logically an $n$-cell, is it possible that $S \setminus E$ is nonempty and $p(x) > 2$ and
constant on $S \setminus E$?

It will be shown that in a slightly more general situation than
that of the above problem, it necessarily follows that $p(x) = 2$ on
$S \setminus E$.

The following notation will be used throughout this paper. For a
semigroup $S$, $E = \{ x : x \in S, x^2 = x \}$ and for $e \in E$, $H(e)$ is the maximal
subgroup of $S$ containing the idempotent $e$. $H = \bigcup \{ H(e) : e \in E \}$ and
functions $\gamma$ and $\theta$ are defined as in [5], that is, for $x \in H$, $\gamma(x)$ is the
idempotent of the unique maximal subgroup to which $x$ belongs and
$\theta(x)$ is the inverse of $x$ in this group.

The following theorem will be proved:

Theorem. Let $S$ be a compact semigroup with the properties:
(1) $S = H$,
(2) for $e \in E$, $H(e)$ is totally disconnected.

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(3) there exists a closed subset $B$ of $S$ such that:
(a) $S \setminus B$ is a connected manifold dense in $S$,
(b) $\dim B < \dim S$.

Then $S$ is a pointwise periodic semigroup and $p(x) \leq 2$ for each $x$ in $S$.

Before proving this theorem the following lemma will be proved.

**Lemma.** Let $S$ be a compact semigroup such that $S = H$ and for each $e$ in $E$, $H(e)$ is totally disconnected. Then $\dim S = \dim E$.

**Proof.** Let the equivalence relation $\mathcal{E}$ be defined as in [5]. Since $S = H$, each $\mathcal{E}$-class is a group and $S/\mathcal{E}$ is homeomorphic to $E$ by the restriction to $E$ of the canonical map of $S$ onto $S/\mathcal{E}$. By Anderson and Hunter [1, Lemma 5, p. 254], it follows that

\[ \dim S \leq \dim S/\mathcal{E} + \max \{ \dim H(e) : e \in E \}. \]

By assumption each $H(e)$ is totally disconnected, hence zero dimensional. Thus $\dim S \leq \dim S/\mathcal{E} = \dim E$ and since $E \subseteq S$, it follows that $\dim S = \dim E$.

**Proof of Theorem.** If it can be shown that $\theta(x) = x$ for each $x$ in $S \setminus B$, then it would follow that $x^3 = x \theta(x) x = x y(x) x$ for $x \in S \setminus B$. But $T = \{ x : x^3 = x \}$ is a closed subset of $S$ and if $S \setminus B \subseteq T$, then $S = T$ since $S \setminus B$ is assumed to be dense in $S$. Thus it suffices to show that $\theta(x) = x$ for $x$ in $S \setminus B$.

To prove $\theta|S \setminus B$ is the identity, let $A = \theta(S \setminus B) \cup S \setminus B$ and let $\theta_0 = \theta|A$. Since $\theta^2 = \theta$ and $\theta$ is a homeomorphism of $S$ onto $S$ [5], $\theta_0(A) = A$ is an open subset of $S$. Now let $E_0 = E \cap S \setminus B$. By the above lemma, $\dim S = \dim E$ which implies that $\dim E_0 = \dim S = \dim A$, since $\dim B < \dim S$. Hence $E_0$ is a nonempty subset of $S \setminus B$, $\theta_0(E_0) = E_0$ so that $E_0 \subseteq S \setminus B \cap \theta(S \setminus B)$ and $A$ is connected. It will now be shown that $F$, the fixed point set of $\theta_0$, has an interior point. From above, $E_0$ is a subset of the manifold $A$ and $\dim E_0 = \dim A$ so that $E_0$ contains an interior point. Since $E_0 \subseteq F$, $F$ also has an interior point. Because $A$ is a connected manifold, $\theta_0$ is a periodic homeomorphism and $F$ has an interior point, it follows from a theorem of Montgomery and Zippin [3, Theorem 1, p. 223] that $\theta_0$ is the identity map. This completes the proof of the theorem.

**Corollary.** Let $S$ be a pointwise periodic semigroup on an $n$-cell. Then $p(x) \leq 2$ for all $x$ in $S$.

**Proof.** A pointwise periodic semigroup is the union of groups [2, Theorem 1.9, p. 20], hence $S = H$. Also $H(e)$ is totally disconnected for each $e$ in $E$ since $H(e)$ is a compact periodic group. Letting $B$, in
the theorem, be the bounding \((n-1)\)-sphere of \(S\), the corollary follows immediately.

**Example (Wallace [6])**. Let \(I = [0, 1]\) be the unit interval with multiplication \(xy = \min\{x, y\}\). Let \(G = \{e, a\}\) be the two element group with identity \(e\) and let \(T_0 = G \times I\) with coordinatewise multiplication. Then \(K_0 = G \times \{0\}\) is an ideal of \(T_0\) and \(T = T_0/K_0\), the Rees quotient of \(T_0\) by the ideal \(K_0\) is a pointwise periodic semigroup whose topological space is a 1-cell.

\(T^n\), the \(n\)-fold cartesian product of \(T\), with coordinatewise multiplication is a pointwise periodic semigroup on an \(n\)-cell.

The above example may be varied by letting one of the intervals in the cartesian product be a semigroup with multiplication \(xy = x\). In this manner one obtains a pointwise periodic semigroup without a two-sided identity.

**References**


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