HOMOTOPOICAL NILPOTENCE OF S³

GERALD J. PORTER

In [1] Berstein and Ganea define the nilpotence of an $H$-space to be the least integer $n$ such that the $n$-commutator is nullhomotopic. We prove that $S³$ with the usual multiplication is 4 nilpotent.

Let $X$ be an $H$-space. The 2-commutator $c_2: X \times X \to X$ is defined by $c_2(x, y) = x y x^{-1} y^{-1}$ where the multiplication and inverses are given by the $H$-space structure of $X$. The $n$-commutator $c_n: X^n \to X$ is defined inductively by $c_n = c_2(c_{n-1})$.

Let $T^n(X)$ denote the subset of $X^n$ consisting of those $n$-tuples $(x_1, \ldots, x_n)$ such that $x_i = *$ (the base point) for at least one $i$. It is well known that $c_n \mid T^n(X) \sim *$. Thus a map $\phi_n: X^n / T^n(X) \to X$ may be defined such that the homotopy class of $\phi_n$ depends only upon the homotopy class of $c_n$. Let $\Phi_n$ be the homotopy class of $\phi_n$. $\Phi_n$ is the Samelson product and $c_n$ is nullhomotopic if and only if $\Phi_n = 0$.

The usual multiplication for $S³$ is that obtained by considering $S³$ to be the set of unit quaternions. With this multiplication $Q_\infty$, infinite quaternionic projective space, is a classifying space for $S³$.

Let $T: \pi(\Sigma S^n, X) \to \pi(S^n, \Omega X)$ be defined by $(Tf(s))(t) = f(t, s)$, where $s \in S^n, f \in \pi(\Sigma S^n, X)$, and $t \in I$. $T$ is an isomorphism of the homotopy groups.

Samelson [3] has shown that if $j: S^1 \to Q^n$ is inclusion, $Tj: S³ \to \Omega Q_\infty$ is an $H$-homomorphism which is also a homotopy equivalence. He uses this to show that $T[[[j, j], j], j] = (Tj)_* \Phi_3$ where the product on the left is the 3-fold iterated Whitehead product. Since $T$ is an isomorphism, to show that $S³$ is 4 nilpotent it suffices to show that the four-fold iterated Whitehead product of $j$ is zero and the three-fold product is nonzero.

Let $i_4$ be the identity map on $S^4$. Hilton [2] has shown that $0 \neq [[[i_4, i_4], i_4] \in \pi_10(S^9)$ is the image of an element in $\pi_0(S^9)$ under the suspension homomorphism. He uses this fact to prove $[[[i_4, i_4], i_4], i_4] = 0$.

**Lemma 1.** $[[[j, j], j], j] = 0$.

**Proof.** $[[[j, j], j], j] = j_* [[[i_4, i_4], i_4], i_4] = 0$.

**Lemma 2.** $[[j, j], j] \neq 0$.

Received by the editors May 3, 1963.
Proof. Consider the following diagram:

\[
\begin{array}{c}
\pi_{10}(Q_\infty) 
\xrightarrow{T} \pi_9(\Omega Q_\infty) \\
\uparrow j^* \hspace{1cm} \uparrow (Tj)^* \\
\pi_{10}(S^4) \xleftarrow{\Sigma} \pi_9(S^3).
\end{array}
\]

Let \( f \in \pi_9(S^3) \), \( s \in S^3 \) and \( t \in I \) then

\[
(Tj_*\Sigma f(s))(t) = j_*\Sigma f(t, s) = j(\Sigma f)(t, s) = j(t, f(s)) = (Tj)_*f(s)(t).
\]

Thus the above diagram commutes, i.e., \((Tj)_* = Tj_*\Sigma\). Since \( Tj \) is a homotopy equivalence \((Tj)_* \) is an isomorphism. By a remark above there is a \( g \in \pi_9(S^3) \) such that \( \Sigma g = [[i_4, i_4], i_4] \). We thus have \( 0 \neq (Tj)_*g = Tj_*\Sigma g = T[[j, j], j] \). Therefore \( [[j, j], j] \neq 0 \).

Theorem. \( S^3 \) with the usual multiplication is 4 homotopy nilpotent.

Corollary (to the proof). \( \Sigma \Phi_3 = [[i_4, i_4], i_4] \).

References


Cornell University