HOMOTOPICAL NILPOTENCE OF $S^3$

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In [1] Berstein and Ganea define the nilpotence of an $H$-space to be the least integer $n$ such that the $n$-commutator is nullhomotopic. We prove that $S^3$ with the usual multiplication is 4 nilpotent.

Let $X$ be an $H$-space. The 2-commutator $c_2: X \times X \rightarrow X$ is defined by $c_2(x, y) = xyx^{-1}y^{-1}$ where the multiplication and inverses are given by the $H$-space structure of $X$. The $n$-commutator $c_n: X^n \rightarrow X$ is defined inductively by $c_n = c_2(c_{n-1}X)$.

Let $T^n(X)$ denote the subset of $X^n$ consisting of those $n$-tuples $(x_1, \ldots, x_n)$ such that $x_i = *$ (the base point) for at least one $i$. It is well known that $c_n|T^n(X) \sim *$. Thus a map $\phi_n: X^n/T^n(X) \rightarrow X$ may be defined such that the homotopy class of $\phi_n$ depends only upon the homotopy class of $c_n$. Let $\Phi_n$ be the homotopy class of $\phi_n$. $\Phi_n$ is the Samelson product and $c_n$ is nullhomotopic if and only if $\Phi_n = 0$.

The usual multiplication for $S^3$ is that obtained by considering $S^3$ to be the set of unit quaternions. With this multiplication $Q_\infty$, infinite quaternionic projective space, is a classifying space for $S^3$.

Let $T: \pi(\mathbb{H}S^n, X) \rightarrow \pi(S^n, OX)$ be defined by $(Tf(s))(t) = f(t, s)$, where $s \in S^n, f \in \pi(\mathbb{H}S^n, X)$, and $t \in I$. $T$ is an isomorphism of the homotopy groups.

Samelson [3] has shown that if $j: S^1 \rightarrow Q^n$ is inclusion, $Tj: S^3 \rightarrow \Omega Q_\infty$ is an $H$-homomorphism which is also a homotopy equivalence. He uses this to show that $T([j, j], j) = (Tj)_* \Phi_s$ where the product on the left is the 3-fold iterated Whitehead product. Since $T$ is an isomorphism, to show that $S^3$ is 4 nilpotent it suffices to show that the four-fold iterated Whitehead product of $j$ is zero and the three-fold product is nonzero.

Let $i_4$ be the identity map on $S^4$. Hilton [2] has shown that $0 \neq [[i_4, i_4], i_4] \in \pi_{10}(S^4)$ is the image of an element in $\pi_9(S^3)$ under the suspension homomorphism. He uses this fact to prove $[[[i_4, i_4], i_4], i_4] = 0$.

**Lemma 1.** $[[[j, j], j], j] = 0$.

**Proof.** $[[[j, j], j], j] = j_*[[[i_4, i_4], i_4], i_4] = 0$.

**Lemma 2.** $[[j, j], j] \neq 0$.

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Proof. Consider the following diagram:

\[ \pi_{10}(Q_\infty) \stackrel{T}{\to} \pi_9(\Omega Q_\infty) \]
\[ \uparrow j_* \quad \uparrow (Tj)_* \]
\[ \pi_{10}(S^4) \leftarrow \pi_9(S^3). \]

Let \( f \in \pi_9(S^3) \), \( s \in S^3 \) and \( t \in I \) then

\[ (Tj_* \Sigma f(s))(t) = j_* \Sigma f(t, s) = j(\Sigma f)(t, s) = j(t, f(s)) = (Tj)_* f(s)(t). \]

Thus the above diagram commutes, i.e., \( (Tj)_* = Tj_* \Sigma \). Since \( Tj \) is a homotopy equivalence \( (Tj)_* \) is an isomorphism. By a remark above there is a \( g \in \pi_9(S^3) \) such that \( \Sigma g = [[[i_4, i_4], i_4]. \) We thus have \( 0 \neq (Tj)_* g = Tj_* \Sigma g = T[[j, j], j]. \) Therefore \( [[[j, j], j] \neq 0. \)

**Theorem.** \( S^3 \) with the usual multiplication is 4 homotopy nilpotent.

**Corollary (to the proof).** \( \Sigma \Phi_4 = [[[i_4, i_4], i_4]. \)

**References**


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