ON THE NEWTON POLYTOPE

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1. Introduction. The theory of the Newton polygon of a polynomial in one variable with coefficients in a complete non-Archimedean valued field is well known (see, for example, [1], [2], [3], [6]). In [4], Krasner states that one may construct an analogous Newton polytope for a polynomial in several variables. In this paper we explore the properties of the Newton polytope.

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2. Preliminaries. Let $K$ be a complete field with respect to a non-Archimedean rank one valuation $x \rightarrow \text{ord } x$ with value group $\Theta \subset \mathbb{R}$, where $\mathbb{R}$ denotes the additive group of real numbers. We shall assume that $\Theta$ is dense in $\mathbb{R}$. Let $\mathbb{F}$ be the algebraic closure of $K$, and extend the valuation to $\mathbb{F}$ in the natural manner. As in [2], for each real number $b$ we define $r_b = \{ x \in \mathbb{F} : \text{ord } x = b \}$. 

Definition 1. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in K[x]$. For any $p \in \mathbb{R}$, $v(f; p) = \text{Min}_{0 \leq i \leq n} (\text{ord } a_i + ip)$. 

Remark. $v(f; p)$ is the $Y$-intercept of the lower line of support of the Newton polygon of $f$ with slope $-p$.

We need the following results from the one-variable theory.

Proposition 1. Let $f(x) \in K[x]$ have a zero on $\Gamma_r$. Then for any $\lambda \in \Theta$ satisfying the inequality $\lambda \geq v(f; r)$, there exists $x \in \mathbb{F}$ such that $\text{ord } f(x) = \lambda$.

Proof. (a) If $a_0 \neq 0$ and $-r$ is the slope of the first side of the Newton polygon of $f$ (i.e., if, for all $r' > r$, $f$ has no zero on $\Gamma_{r'}$) then clearly $v(f; r) = \text{ord } a_0$. Therefore, we need only choose $\gamma \in \Gamma_\lambda$ such that $\text{ord } (a_0 - \gamma) = \text{ord } a_0$, for then the polynomials $f(x)$ and $f(x) - \gamma$ will have identical Newton polygons. If $\lambda > v(f; r)$, then for any $\gamma \in \Gamma_\lambda$, $\text{ord } (a_0 - \gamma) = \text{ord } a_0$; if $\lambda = v(f; r)$, we choose $\alpha, \beta \in \Gamma_0$ such that $\alpha + \beta \in \Gamma_0$ (this can be done since the residue class field of $\mathbb{F}$ contains more than two elements), and put $\gamma = a_0(1 + a \beta^{-1})$.

(b) If either $a_0 = 0$ or $-r$ is not the slope of the first side of the Newton polygon of $f$, let $\gamma$ be any element of $\Gamma_\lambda$, and consider the Newton diagram of $f(x) - \gamma$: clearly the Newton diagram of $f(x) - \gamma$ coincides with the Newton diagram of $f(x)$, with the possible excep-
tion of the points with zero abscissa. Since \( \text{ord}(a_0 - \gamma) \geq v(f; r) \), \(-r\) is the slope of a side of the Newton polygon of \( f(x) - \gamma \).

The following result is essentially identical to Lemma 1.2 of [2].

**Lemma 1.** Let \( f_1(x), f_2(x), \ldots, f_n(x) \) be a finite set of polynomials with coefficients in \( K \), let \( \rho \in \mathcal{R} \). Then there exists \( \xi \in \Gamma_\rho \) such that \( \text{ord} f_i(\xi) = v(f_i; \rho) \), \( i = 1, 2, \ldots, n \).

**Proof.** Let \( v(f_i; \rho) = M_i \), \( i = 1, 2, \ldots, n \); then \( M_i \in \mathcal{R} \). Therefore, we may choose \( \pi_i \in \Gamma_{M_i}, \pi \in \Gamma_\rho \). For each \( i \), we put \( g_i(x) = f_i(\pi x)/\pi \). Then the coefficients of \( g_i(x) \) are integral and the image of \( g_i(x) \) in the residue class field of \( \mathcal{R} \) is nontrivial. Since the residue class field is infinite there is a unit \( \xi' \) in \( \mathcal{R} \) such that \( \text{ord} g_i(\xi') = 0, i = 1, 2, \ldots, n \). If we put \( \xi = \pi \xi' \), we have the desired result.

3. **The Newton polytope.** Let \( f(x, y) = \sum a_{ij} x^i y^j \in K[x, y] \). The point set \( \{(i, j, \text{ord} a_{ij})\} \) is called the Newton diagram of \( f(x, y) \). We define the convex closure of the Newton diagram of \( f(x, y) \) with the point \((0, 0, +\infty)\) to be the Newton polytope of \( f(x, y) \).

**Remark.** The Newton polytope of \( f(x, y) \) is the graph of the function

\[
\Pi_f(X, Y) = \sup_{\mu, \nu \in \mathcal{R}} [v(f; \mu, \nu) - \mu X - \nu Y],
\]

where \( v(f; \mu, \nu) \) is defined in the obvious manner generalizing Definition 1: \( v(f; \mu, \nu) = \min_{i, j} (\text{ord} a_{ij} + i\mu + j\nu) \) (see [5, p. 49]).

Let \((\xi, \eta) \in \mathcal{R} \times \mathcal{R} \), suppose \((\xi, \eta) \in \Gamma_{\rho} \times \Gamma_\sigma \). The following result gives an estimate for \( \text{ord} f(\xi, \eta) \) in terms of \( \rho, \sigma \).

**Proposition 2.** Let \( P \) be the lower plane of support of the Newton polytope of \( f(x, y) \), with \( \partial Z/\partial X = -\rho, \partial Z/\partial Y = -\sigma \). Suppose \((\xi, \eta) \in \Gamma_{\rho} \times \Gamma_\sigma \). If only one vertex of the polytope lies on \( P \), then only one term of \( f(\xi, \eta) \) attains minimal order, and then \( \text{ord} f(\xi, \eta) = v(f; \rho, \sigma) \), the Z-axis intercept of \( P \). Otherwise, \( \text{ord} f(\xi, \eta) \geq v(f; \rho, \sigma) \).

**Proof.** Let the plane \( P_{ij} \) be defined by the equation \( Z + \rho X + \sigma Y = \text{ord}(a_{ij} \xi^i \eta^j) \). Then the point \((i, j, \text{ord} a_{ij}) \) in the Newton diagram of \( f(x, y) \) lies in \( P_{ij} \); but ord \((a_{ij} \xi^i \eta^j) < \text{ord}(a_{i'j'} \xi^{i'} \eta^{j'}) \) (respectively \( \text{ord}(a_{ij} \xi^i \eta^j) \leq \text{ord}(a_{i'j'} \xi^{i'} \eta^{j'}) \)) if and only if the intercept cut off on the Z-axis by the plane \( P_{ij} \) is less than (respectively less than or equal to) that cut off by \( P_{i'j'} \). Thus, \( \text{ord}(a_{i'j'} \xi^{i'} \eta^{j'}) = \min_{i, j} \text{ord}(a_{ij} \xi^i \eta^j) \) if and only if \( P_{i'j'} \) is the lower plane of support of the Newton polytope with \( \partial Z/\partial X = -\text{ord} \xi, \partial Z/\partial Y = -\text{ord} \eta \).

**Corollary.** If \((\xi, \eta) \) is a zero of \( f(x, y) \), then the lower plane of sup-
port $P$ of the Newton polytope of $f(x, y)$ with $\partial Z/\partial X = -\text{ord } \xi, \partial Z/\partial Y = -\text{ord } \eta$ contains an edge of the polytope.

**Remark.** No distinction is made here between the plane $P$ containing an edge or a face of the polytope.

The converse to the corollary of Proposition 2 is also valid. Thus, the Newton polytope of $f(x, y)$ gives an explicit criterion for determining the existence of a zero of $f(x, y)$ on $\Gamma_r \times \Gamma_s$. Before proceeding to the proof of the converse, we introduce the following notation.

Let $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n$, $f_i(x) \in K[x]$, $i = 0, 1, 2, \ldots, n$. We shall assume that $f(x, y) \notin K[x]$, $f_n(x) \neq 0$. Let $\Pi$ denote the Newton polytope of $f(x, y)$. For $\rho \in \mathcal{O}$, let $\Lambda_\rho$ be the convex closure in the $YZ$-plane of the point set $\{(0, j, \text{ord } f_j(\xi)) : j = 0, 1, 2, \ldots, n\}$ with the point $(0, 0, +\infty)$. For $\xi \in K$, let $\Lambda_\xi$ be the convex closure in the $YZ$-plane of the point set $\{(0, j, \text{ord } f_j(\xi)) : j = 0, 1, 2, \ldots, n\}$ with the point $(0, 0, +\infty)$. We observe that $\Lambda_\xi$ is the Newton polygon of the polynomial $g_\xi(y) = \sum f_j(\xi)y^j$, and that if ord $\xi = \rho$, then no point of $\Lambda_\rho$ lies below $\Lambda_\xi$. Let $\Pi_j$ denote the Newton polygon of the polynomial $f_j(x)$ in the plane $Y = j$, and finally let $l_j(\rho)$ be the lower line of support of $\Pi_j$ with slope $-\rho$ in the plane $Y = j$.

**Proposition 3.** Let $f(x, y) \in K[x, y]$, let $r, s \in \mathcal{O}$. Suppose $P_r$ is the lower plane of support of $\Pi$, the Newton polytope of $f(x, y)$, with equation $Z + rX + sY + d = 0$. If $P_r$ contains an edge of $\Pi$, then there is a point $(\xi, \eta) \in \Gamma_r \times \Gamma_s$ such that $f(\xi, \eta) = 0$.

**Proof.** Suppose $P_r$ contains an edge of $\Pi$ with direction numbers $(\alpha, \beta, \gamma)$. Since $P_r$ cannot contain a vertical line, either $\alpha$ or $\beta$ is different from zero. We may assume, with no loss of generality, that $\beta \neq 0$. Then a pair of points $p_1 = (i_1, j_1, \text{ord } a_{i_1j_1})$, $p_2 = (i_2, j_2, \text{ord } a_{i_2j_2})$ of the Newton diagram of $f(x, y)$ is on $P_r$, with $j_1 \neq j_2$. Since $P_r$ is a lower plane of support of $\Pi$ containing $p_1$ and $p_2$, with $\partial Z/\partial X = -r$, it follows that $l_{i_1}(r)$ and $l_{i_2}(r)$ are in $P_r$. By Lemma 1, we may choose $\xi \in \Gamma_r$ such that $\text{ord } f_{j_2}(\xi) = \nu(f_{j_2}, r)$, $\text{ord } f_{j_1}(\xi) = \nu(f_{j_1}, r)$. Thus, the points $q_1 = (0, j_1, \text{ord } f_{j_1}(\xi))$ and $q_2 = (0, j_2, \text{ord } f_{j_2}(\xi))$ of the Newton diagram of $g_\xi(y)$ are in $P_r$, and are therefore on a side of $\Lambda_\xi$ which lies in $P_r$ (since no point of $\Lambda_\xi$ can lie below the intersection of $P_r$ with the $YZ$-plane). But since $\Lambda_\xi$ lies in the $(X = 0)$-plane, we see that the side of $\Lambda_\xi$ determined by $q_1, q_2$ has slope $\partial Z/\partial Y = -s$; therefore, the polynomial $g_\xi(y)$ has a root $\eta \in \Gamma_s$. Hence, $(\xi, \eta) \in \Gamma_r \times \Gamma_s$ and $f(\xi, \eta) = 0$.

We summarize these results in
Theorem 1. Let \( f(x, y) \in \mathbb{K}[x, y] \), let \( r, s \in \mathbb{R} \), and let \( P_r \) be the lower plane of support of the Newton polytope of \( f(x, y) \) with \( \frac{\partial Z}{\partial X} = -r \), \( \frac{\partial Z}{\partial Y} = -s \). There is a zero \((\xi, \eta)\) of \( f(x, y) \) such that \( \text{ord } \xi = -r \), \( \text{ord } \eta = -s \), if, and only if, the plane \( P_r \) contains an edge of the polytope.

4. Distinguished values.

Definition 2. Let \( D \) be a subset of \( \mathbb{K} \times \mathbb{R} \), let \( r \) (respectively \( s \)) be a real number. We say that \( r \) is \( x \)-distinguished on \( D \) (respectively, \( s \) is \( y \)-distinguished on \( D \)) if there are infinitely many \( s \in \mathbb{R} \) (respectively, infinitely many \( r \in \mathbb{R} \)) such that \( D \cap (\Gamma_r \times \Gamma_s) \neq \emptyset \).

Proposition 4. Let \( f(x, y) \in \mathbb{K}[x, y] \), suppose \( f(x, y) \neq 0 \); let \( D = V(f) = \{(x, y) \in \mathbb{K} \times \mathbb{R} : f(x, y) = 0\} \). The set of real numbers which are \( x \)-distinguished on \( D \) (respectively, \( y \)-distinguished on \( D \)) is finite.

Proof. Let \( f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n \in \mathbb{K}[x] \), \( 0 \leq i \leq n \). Since \( f(x, y) \neq 0 \), not all the polynomials \( \{f_i(x)\} \) are zero. Let \( \mathbb{B} \) be the subset of \( \mathbb{B} \{f_i(x) : 0 \leq i \leq n\} \) consisting of those polynomials which are nonzero, and let \( \mathbb{R} \) be the set of values of zeros of polynomials in \( \mathbb{B} \), i.e., \( r \in \mathbb{R} \) if there is a pair \((f, \xi) \in \mathbb{B} \times \Gamma_s \) such that \( f(\xi) = 0 \). Clearly \( \mathbb{R} \) is a finite set. Suppose \( r' \in \mathbb{R} \). Then for \( \xi \in \Gamma_r \), the Newton diagram of \( g_\xi(y) = f(\xi, y) \) depends only on \( \text{ord } \xi \). Therefore, as \( \xi \) runs through \( \Gamma_r \), there is only a finite number of \( s \in \mathbb{R} \) such that \( g_\xi \) has a zero on \( \Gamma_s \). Therefore if \( r' \in \mathbb{R} \), \( r' \) is not \( x \)-distinguished on \( D \).

The set of real numbers which are distinguished for a given polynomial is determined by the Newton polytope of that polynomial. In fact, we shall prove

Theorem 2. Let \( f(x, y) \in \mathbb{K}[x, y] \), suppose \( f(0, 0) \neq 0 \). Then \( \rho \) is \( x \)-distinguished on \( V(f) \) if, and only if, there is an edge of the Newton polytope of \( f(x, y) \) with direction numbers \((1, 0, -\rho)\).

The proof of Theorem 2 will be a trivial consequence of Propositions 5 and 6.

Lemma 2. Let \( f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n \in \mathbb{K}[x, y] \), suppose \( f(x, y) \in \mathbb{K}[x] \), \( f(0, 0) \neq 0 \). Let \( \rho \in \mathbb{R} \), and let \( \Lambda_s, \Pi_i, 0 \leq j \leq n \), be as previously defined. If the point \((0, j_0, v(f_{j_0}; \rho))\) is on \( \Lambda_s \), then there is a vertex \((i_0, j_0, \text{ord } a_{i_0j_0}) \) of \( \Pi_{j_0} \) which is on the Newton polytope of \( f(x, y) \).

Proof. Suppose the point \((0, j_0, v(f_{j_0}; \rho))\) is on the side of \( \Lambda_s \) with vertices \((0, j_1, v(f_{j_1}; \rho)), (0, j_2, v(f_{j_2}; \rho)) \), and suppose \( j_1 < j_2 \). Let \( P \) be the plane determined by the (parallel) lines \( l_{j_1}(\rho) \) and \( l_{j_2}(\rho) \). Then certainly \( l_{j_2}(\rho) \) lies in \( P \). It remains only to be shown that \( P \) is a
lower plane of support of the polytope. Suppose not; then there is a point \((i', j', \text{ord} a_{i'j'})\) below \(P\). Hence \((0, j', v(f_{j'; \rho}))\) lies below the line \(P \cap (X = 0)\). This contradicts convexity of \(\Lambda_\rho\) in the \(YZ\)-plane.

**Corollary.** Using the above notation, if \((0, j_0, v(f_{j_0}; \rho))\) is on \(\Lambda_\rho\), and if \(\text{ord} f_\xi(\xi)\) has more than one value for \(\xi \in \Gamma_\rho\), then a side of \(\Pi_{j_0}\) is on the polytope of \(f(x, y)\).

**Proposition 5.** If \(\rho \in \Theta\) is \(x\)-distinguished on \(V(f)\), then there is an edge of the Newton polytope of \(f(x, y)\) with direction numbers \((1, 0, -\rho)\).

**Proof.** For \(\xi \in \Gamma_\rho\), we let \(g_\xi(y)\), \(\Lambda_\xi\), \(\Lambda_\rho\) be defined as before. Since \(\rho\) is \(x\)-distinguished on \(V(f)\), the set of slopes of sides of the polygons \(\{\Lambda_\xi: \xi \in \Gamma_\rho\}\) is infinite. Consider the set of non-negative integers \(j\) with the property that \((0, j, v(f_j; \rho))\) is a vertex of \(\Lambda_\rho\) and \(\{\text{ord} f_\xi(\xi): \xi \in \Gamma_\rho\}\) has more than one element. If this set were empty, it would follow that \(\Lambda_\rho = \Lambda_\xi\) for each \(\xi \in \Gamma_\rho\), contradicting the hypothesis. Let \(j_0\) denote the smallest integer of this set.

By the previous corollary, \(l_{j_0}(\rho)\) contains a side of \(\Pi_{j_0}\) and this side is on \(\Pi\). To complete the proof of Proposition 5, we need only show that this side of \(\Pi_{j_0}\) is indeed an edge of the polytope. If \(j_0\) is either 0 or \(n\), this is certainly the case. Otherwise, we may choose integers \(j_1, j_2\) such that \((0, j_1, v(f_{j_1}; \rho))\), \((0, j_0, v(f_{j_0}; \rho))\), and \((0, j_2, v(f_{j_2}; \rho))\) are distinct adjacent vertices of \(\Lambda_\rho\), with \(0 \leq j_1 < j_0 < j_2 \leq n\). Let \(P_1\) be the plane determined by the lines \(l_{j_0}(\rho), l_{j_1}(\rho)\), and let \(P_2\) be the plane determined by the lines \(l_{j_0}(\rho), l_{j_2}(\rho)\). By the concluding argument of Lemma 2, \(P_1\) and \(P_2\) are lower planes of support of the Newton polytope of \(f(x, y)\). By choice of \(j_1\) and \(j_2\), they are distinct, and their intersection is the line \(l_{j_0}(\rho)\). This completes the proof.

**Proposition 6.** If there is an edge of the Newton polytope of \(f(x, y)\) with direction numbers \((1, 0, -\rho)\), then \(\rho \in \Theta\) and \(\rho\) is \(x\)-distinguished on \(V(f)\).

**Note.** It is not necessary to assume here that \(\rho \in \Theta\).

**Proof.** We again write \(f(x, y) = \sum_{j=0}^n f_j(x)y^j\); what we are required to show is that, if there is a polynomial \(f_\xi(x)\) such that \(f_\xi(x)\) has a zero on \(\Gamma_\rho\) and, moreover, that the side of \(\Pi_\xi\) of slope \(-\rho\) is an edge of the Newton polytope \(\Pi\) of \(f(x, y)\), then \(\rho\) is \(x\)-distinguished on \(V(f)\). That is, we must show that the set \(\xi_\rho = \{\lambda: -\lambda\) is the slope of a side of \(\Lambda_\xi\), for some \(\xi \in \Gamma_\rho\}\) is infinite. (We observe that \(\rho \in \Theta\), from the one-variable Newton polygon theory applied to \(f_\xi(x)\).)

**Case 1.** For some \(k, 1 \leq k \leq n, f_k\) has no zeros on \(\Gamma_\rho\). Let \(k_0\) be the smallest such \(k\). Then either \(k_0 = 0\) or \(k_0 > 0\).
(1a) Suppose \( k_0 = 0 \). Let \( i_0 \) be the smallest integer with the property that a side of \( \Pi_{i_0} \) of slope \(-\rho\) is an edge of \( \Pi \). Then \((0, i_0, v(f_{i_0}; \rho))\) is a vertex of \( \Lambda_\rho \). Moreover, \( i_0 > 0 \), since \( f_0 \) has no zeros on \( \Gamma_\rho \).

Let the vertices of \( \Lambda_\rho \) in the \( YZ \)-plane have \( Y \)-coordinates \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_i \), let \( i_0 = \alpha_0 \). Then for all \( \xi \in \Gamma_\rho \), the polygons \( \Lambda_i \) and \( \Lambda_\rho \) agree in vertices whose \( Y \)-coordinates are \( \alpha_0, \alpha_1, \cdots, \alpha_{i-1} \).

Consider the set \( \mathcal{B} \) of \( Z \)-coordinates of points on the line \( \overline{EG} \) in Figure 1 which are also in \( \mathcal{S} \). Since \( \mathcal{S} \) is dense in \( R \), the set \( \mathcal{B} \) is infinite. But \( E \) has coordinates \((i_0, v(f_{i_0}; \rho))\), whence from Proposition 1, we may choose, for each \( r \in \mathcal{B} \), an element \( \xi \in \Gamma_\rho \) such that \( \text{ord} \, f_{i_0}(\xi) = r \). Let \( \mathcal{Z} \) be a set of representatives of \( \mathcal{B} \) in \( \Gamma_\rho \): if \( \xi \in \mathcal{Z} \) then \( \text{ord} \, f_{i_0}(\xi) \in \mathcal{B} \), and \( \xi, \xi' \in \mathcal{Z} \), \( \xi \neq \xi' \) implies \( \text{ord} \, f_{i_0}(\xi) \neq \text{ord} \, f_{i_0}(\xi') \). The definition of \( \mathcal{B} \) guarantees that \((\alpha_0, \text{ord} \, f_{i_0}(\xi))\) is a vertex of \( \Lambda_i \) for any \( \xi \in \mathcal{Z} \).

For \( \alpha_{i-1} \leq \nu \leq \alpha_i - 1 \), we define the sets \( \mathcal{E}_\nu^{(1)} = \{ \xi \in \mathcal{Z}: (\nu, \text{ord} \, f_{i_0}(\xi)), (i_0, \text{ord} \, f_{i_0}(\xi)) \} \) are vertices of a side of \( \Lambda_i \). Then, since \( \mathcal{Z} = \bigcup_{\alpha_{i-1} < \nu < \alpha_i} \mathcal{E}_\nu^{(1)} \), we may choose \( \nu_1 \) to be the largest integer with the property that \( \nu_1 < i_0 \) and \( \mathcal{E}_{\nu_1}^{(1)} \) is an infinite set. Let \( \mathcal{X}_1 = \{ \text{ord} \, f_{i_0}(\xi), \xi \in \mathcal{E}_{\nu_1}^{(1)} \}. \) If \( \mathcal{X}_1 \) is finite, then the set \{ \text{ord} \, f_{i_0}(\xi) - \text{ord} \, f_{i_0}(\xi), \xi \in \mathcal{E}_{\nu_1}^{(1)} \} \) is infinite, whence so is \( \mathcal{B} \). Otherwise we define, for \( \alpha_{i-1} < \nu < \nu_1 \), the sets \( \mathcal{E}_{\nu}^{(2)} = \{ \xi \in \mathcal{E}_{\nu_1}^{(1)}: (\nu, \text{ord} \, f_{i_0}(\xi)), (\nu_1, \text{ord} \, f_{i_0}(\xi)) \} \) are vertices of a side of \( \Lambda_i \}, \) and choose \( \nu_2 \) to be the largest integer with the property that \( \nu_2 < \nu_1 \) and \( \mathcal{E}_{\nu_2}^{(2)} \) is an infinite set. We then define \( \mathcal{X}_2 = \{ \text{ord} \, f_{i_0}(\xi), \xi \in \mathcal{E}_{\nu_2}^{(2)} \}. \) Proceeding in this manner, we define a sequence of integers \( \nu_1 > \nu_2 > \cdots > \nu_m \geq \alpha_{i-1} \), and a corresponding sequence of sets \( \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m \). If \( \mathcal{X}_i \) is finite for some \( i \), then (1a) is proved. Otherwise, we may assume that \( m \) is such that \( \nu_m = \alpha_{i-1} \); but then, \( \mathcal{X}_m = \{ \text{ord} \, f_{i_0, \nu_m}(\xi), \xi \in \mathcal{E}_{\nu_m}^{(m)} \} = \{ v(f_{\alpha_{i-1}}; \rho) \} \), which is certainly finite. Thus, case (1a) is proved.

\[ \text{If } \alpha_1 = m, \text{ let } \mathcal{B} = \{ \lambda \in \mathcal{S}: \text{ord} \lambda \geq v(f_m; \rho) \}. \]
(1b) Suppose $k_0 > 0$. Then $f_\rho$ has a zero on $\Gamma_\rho$. Hence, by Proposition 1, we may choose a sequence $\{\xi_h\} \subset \Gamma_\rho$ such that $\text{ord}_h f_\rho(\xi_h) \to \infty$ as $h \to \infty$. If $\rho$ is not $x$-distinguished on $V(f)$, we must then have $\text{ord}_h f_\rho(\xi_h) \to \infty$ as $h \to \infty$, $\cdots$, $\text{ord}_h f_{k_0-1}(\xi_h) \to \infty$ as $h \to \infty$. But then, for $h$ sufficiently large, the point $(k_0, \text{ord}_h f_{k_0}(\xi_h)) = (k_0, \nu(f_{k_0}; \rho))$ is on $\Lambda_{k_0}$.

For $0 \leq \nu < k_0$, we let $\mathcal{E}_\nu = \{h \in \mathbb{Z}: (\nu, \text{ord}_h f_\rho(\xi_h)), (k_0, \text{ord}_h f_{k_0}(\xi_h)) \text{ are vertices of a side of } \Lambda_{k_0}\}$. Then for some $\nu_0$, the set $\mathcal{E}_{\nu_0}$ is infinite, whence the set of slopes

$$\left\{\frac{\nu(f_{k_0}; \rho) - \text{ord}_h f_{\nu_0}(\xi_h)}{k_0 - \nu_0} : \nu \in \mathcal{E}_{\nu_0}\right\}$$

is infinite, whence $\rho$ is $x$-distinguished on $V(f)$.

Case 2. $f_j(x)$ has a zero on $\Gamma_\rho$ for each $j$, $0 \leq j \leq n$. If there is an $\eta \in \Gamma_\rho$ such that $f_j(\eta) = 0$ for each $j$, $0 \leq j \leq n$, then certainly $\rho$ is $x$-distinguished on $V(f)$. Therefore, we may assume that no root of one of the $f_j$ is a root of all the $f_j$.

Let $f_0$ have the zeros $\eta_1, \eta_2, \cdots, \eta_w$ on $\Gamma_\rho$. Let $j_0$ be the smallest integer with the property that, for some $i_0$, $1 \leq i_0 \leq w$, $f_{j_0}(\eta_{i_0}) \neq 0$. Choose a neighborhood $N$ of $\eta_i$ in $\mathbb{R}$ such that $\{\text{ord}_h f_{i}(\xi) : \xi \in N\}$ is bounded. Since the valuation of $\mathbb{R}$ is dense, we may choose a sequence $\{\xi_h\} \subset N$ such that $\text{ord}_h f_{j_0}(\xi_h) \to \infty$, $h \to \infty$, for $k = 0, 1, 2, \cdots, j_0 - 1$. But $\{\text{ord}_h f_{j_0}(\xi_h): h = 1, 2, \cdots\}$ is bounded; therefore, for $h$ sufficiently large, the point $(j_0, \text{ord}_h f_{j_0}(\xi_h))$ is on $\Lambda_{k_0}$.

For $0 \leq \nu < j_0$, let $\mathcal{E}_\nu = \{h \in \mathbb{Z}: (\nu, \text{ord}_h f_{j_0}(\xi_h)), (j_0, \text{ord}_h f_{j_0}(\xi_h)) \text{ are vertices of a side of } \Lambda_{k_0}\}$. Then for some $\nu_0$, $\mathcal{E}_{\nu_0}$ is infinite; therefore, the set of slopes

$$\left\{\frac{\text{ord}_h f_{j_0}(\xi_h) - \text{ord}_h f_{\nu_0}(\xi_h)}{j_0 - \nu_0} : \nu \in \mathcal{E}_{\nu_0}\right\}$$

is infinite, whence $\rho$ is $x$-distinguished on $V(f)$.

References


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