CONCERNING CONTINUOUS IMAGES OF
COMPACT ORDERED SPACES

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It is the purpose of this paper to prove that if each of $X$ and $Y$
is a compact Hausdorff space containing infinitely many points, and
$X \times Y$ is the continuous image of a compact ordered space $L$, then
both $X$ and $Y$ are metrizable. The preceding theorem is a generaliza-
tion of a theorem [1] by Mardešić and Papić, who assume that $X$, $Y$,
and $L$ are also connected. Young, in [3], shows that the Cartesian
product of a "long" interval and a real interval is not the continuous
image of any compact ordered space.

In this paper, the word compact is used in the "finite cover" sense.
The phrase "ordered space" means a totally ordered topological space
with the order topology. A subset $M$ of a topological space is said
to be herditarily separable provided each subset of $M$ is separable.
If $a$ and $b$ are points of an ordered space $L$ and $a < b$, then $[a, b]$
will denote the set of all points $x$ of $L$ such that $a \leq x \leq b$
($a < x < b$), provided there is one; also, $[a, b]$ will be used even in the
case where $a = b$. A subset $M$ of an ordered space $L$ is convex provided
that if $a \in M$, $b \in M$, and $a < b$, then $[a, b] \subseteq M$. If $M$ is a subset
of an ordered space $L$, then $G(M)$ will denote the set of all ordered pairs
$(a, b)$ such that (1) $a \in M$, $b \in M$, and $a < b$, and (2) $\{a, b\} = M \setminus [a, b]$,
provided there is one.

Lemma 0. If $M$ is a compact subset of the ordered space $L$, then the
relative topology of $L$ on $M$ is the same as the order topology on $M$.

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Lemma 1. If $M$ is a nondegenerate, totally disconnected, compact subset of an ordered space $L$, then $M$ is metrizable if and only if $G(M)$ is countable.

Proof. Suppose $M$ is metrizable. Since a compact Hausdorff space is metric if and only if it satisfies the second axiom of countability, there is a countable sequence $I_1, I_2, \cdots$ such that (1) for each $n$, $I_n$ is a convex open subset of $L$, and (2) $I_1 \cdot M, I_2 \cdot M, \cdots$ is a countable basis for $M$. There exists a transformation $T$ from $G(M)$ into the ordered pairs of positive integers such that if $(a, b) \in G(M)$ and $T((a, b)) = (p, q)$, then $I_p \cdot \{a, b\} = a$ and $I_q \cdot \{a, b\} = b$. $T$ is easily seen to be a one-to-one transformation, so $G(M)$ is evidently countable.

Suppose $G(M)$ is countable. Let the elements of $G(M)$ be labeled $(a_1, b_1), (a_2, b_2), \cdots$. Let $H$ denote a collection such that if $h \in H$ if and only if (1) there is a positive integer $i$ such that $h$ is the set of all points of $M$ which precede $b_i$, or $h$ is the set of all points of $M$ which follow $a_i$; or (2) there exist integers $i$ and $j$ such that $h = M \cdot [b_i, a_j]$. $H$ is a countable basis for $M$, so $M$ is metrizable.

Lemma 2. If $M$ is a separable subset of the ordered space $L$, then $M$ is hereditarily separable.

Proof. Suppose $H$ is a subset of $M$. There is a countable set $P_1, P_2, \cdots$ dense in $M$ such that if $P \in M$, then for some integer pair $(i, j)$, $P_i \leq P \leq P_j$. For each integer pair $(i, j)$ such that $P_i \leq P_j$ and $[P_i, P_j] \cdot H$ exists, let $H_{ij}$ denote a countable subset of $[P_i, P_j] \cdot H$ such that if $P \in [P_i, P_j]$, then there exists $R$ in $H_{ij}$ and $S$ in $H_{ij}$ such that $R \leq P \leq S$. $\sum H_{ij}$ is easily seen to be a countable set dense in $H$.

Lemma 3. If $M$ is a nonconnected, separable, compact subset of the ordered space $L$, then $M$ is metrizable if and only if $G(M)$ is countable.

Lemma 4. If the continuous function $f_1$ maps the compact ordered space $K_1$ onto the Hausdorff space $S$, then there is a compact ordered space $K_2$ and a continuous function $f_2$ mapping $K_2$ onto $S$ such that (1) if $K$ is a closed proper subset of $K_2$, then $f_2(K) \neq S$, and (2) if $x$ and $y$ are elements of $K_2$ such that $f_2(x) = f_2(y)$, there is an element $z$ of $K_2$ between $x$ and $y$ such that $f_2(z) \neq f_2(x)$.

Proof. Let $H$ denote the set of closed subsets $m$ of $K$ such that $f_1(m) = S$. Define a partial order $\leq$ on $H$ by saying $m_1 \leq m_2$ if and only if $m_1 \subseteq m_2$. It is easily verified that each chain has a lower bound, so Zorn's lemma applies here, and $H$ has a minimal element $K$. 
For each point $x$ of $K$ let $K_x$ denote the union of all the subsets $k$ of $K$ such that (1) $x \in k$, (2) $k$ is convex relative to $K$, and (3) if $y \in k$, then $f_1(y) = f_1(x)$. Each $K_x$ is closed, and if $x$ and $y$ are elements of $K$, then either $K_x = K_y$ or $K_x \subset K - K_y$. Let $K_2$ denote the set of all $K_x$ for $x \in K$, and suppose $U$ is open in $K_2$ if and only if $U^*$ is open in $K$.

Suppose that $K_2$ is given the natural order induced by the order on $K$, and that $f_2$, which maps $K_2$ onto $S$, is defined by $f_2(K_x) = f_1(x)$. The space $K_2$ and the function $f_2$ satisfy the conclusion of the lemma.

**Lemma 5.** If the continuous function $f$ maps the compact metric space $R$ onto the Hausdorff space $S$, then $S$ is metrizable.

**Proof for the case where $S$ is nondegenerate.** Let $R_1, R_2, \ldots$ denote a countable basis for $R$. Let $T$ denote a collection such that $U \in T$ if and only if there exists two points $s_1$ and $s_2$ of $S$ and a finite integer sequence $j_1, j_2, \ldots, j_n$ such that $f^{-1}(s_1) \subset \sum R_{i,1}, f^{-1}(s_2) \subset R - \sum R_{i,2}$, and $U = S - f(R - \sum R_{i,2})$. The collection $T$ is a countable basis for the compact Hausdorff space $S$, so $S$ is metrizable.

**Lemma 6.** If the continuous function $f$ maps the compact ordered space $K$ onto the infinite, compact Hausdorff space $S$, then there exists a sequence $x_0, x_1, \cdots$ of distinct elements of $S$ such that $x_1, x_2, \cdots$ converges to $x_0$.

**Proof.** Let $y_1, y_2, \cdots$ denote a sequence of distinct elements of $S$, and for each $n$, let $z_n$ denote an element of $f^{-1}(y_n)$. There is an increasing sequence of integers $n_1, n_2, \cdots$ such that $z_{n_1}, z_{n_2}, \cdots$ is monotone, and since $K$ is compact, there is a point $z$ such that the latter sequence converges to $z$. There is a subsequence $j_1, j_2, \cdots$ of $n_1, n_2, \cdots$ such that $f(z_{j_1}) \neq f(z), i = 1, 2, \cdots$. The sequence $x_0, x_1, \cdots$ defined by $x_0 = f(z)$ and $x_i = f(z_{j_i}), i \geq 1$, satisfies the conclusion of the lemma.

**Proof of Theorem.** Suppose that $X$ is not metrizable. Let $u$ denote an element of $X$, and let $g$ map $L$ continuously onto $X \times Y$. Since $\{u\} \times Y$ is the continuous image of a compact ordered space, an application of Lemma 6 yields an infinite sequence of distinct points $(u, b), (u, b_1), (u, b_2), \cdots$, all lying in $\{u\} \times Y$, such that $(u, b_1), (u, b_2), \cdots$ converges to $(u, b)$. The space $Z = \{b, b_1, b_2, \cdots\}$ with the relative topology of $Y$ is an infinite compact Hausdorff space, and the space $X \times Z$ is the continuous image of a compact ordered space, so there exists a compact ordered space $K$ and a continuous function $f$ mapping $K$ onto $X \times Z$ such that the conclusions of Lemma 4 hold.

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* If $U$ is a collection of point sets, then $U^*$ denotes the sum of the sets of the collection $U$. 

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For each positive integer \( n \), let \( H_n \) denote a partition of \( X \times Z \) into the following \( n + 1 \) open and closed sets: \( X \times \{ b_1 \}, X \times \{ b_2 \}, \ldots, X \times \{ b_n \}, X \times \{ b, b_{n+1}, b_{n+2}, \ldots \} \). For each \( n \), let \( K_n \) denote the set of all \( f^{-1}(h) \) for \( h \in H_n \), and let \( I_n \) denote a partition of \( K \) into convex open and closed sets such that if \( I \in I_n \), there is an element \( k \) of \( K_n \) such that \( I \subseteq k \). Since \( K \) is compact, each \( I_n \) is a finite collection. Let \( C \) denote a point set to which a point \( P \) belongs if and only if there is an integer \( n \) and an element \( I \) of \( I_n \) which intersects \( f^{-1}(X \times \{ b \}) \) such that \( P \) is either the right-most point of this intersection or the left-most. \( C \) is a countable set which will be shown to be dense in \( f^{-1}(X \times \{ b \}) \).

Suppose the set \( f^{-1}(X \times \{ b \}) \) contains an open set \( U \). Let \( P \) denote an arbitrary point of \( f^{-1}(X \times \{ b \}) \) and suppose \( f(P) = (x, b) \). For each \( n \) let \( Q_n \) denote an element of \( f^{-1}(x, b_n) \). Some subsequence of the \( Q_i \)'s converges to a point \( Q \) in \( K - U \), and the continuity of \( f \) implies that \( f(Q) = (x, b) \). Therefore, \( f(K - U) = X \times Z \), which is a contradiction. Now suppose that \( P \in f^{-1}(X \times \{ b \}) \) and \( R < P < S \). There is a positive integer \( n \) and a point \( Q \) of \( f^{-1}(X \times \{ b_n \}) \) in \( (R, S) \). Suppose \( P < Q < S \). There is an element \( I \) of \( I_n \) containing \( P \), but not \( Q \), and the right-most point \( T \) of \( I \)-\( f^{-1}(X \times \{ b \}) \) is an element of \( C \) satisfying \( P = T < S \). This case clearly shows why \( C \) is dense in \( f^{-1}(X \times \{ b \}) \).

The separability of \( f^{-1}(X \times \{ b \}) \) implies that \( X \times \{ b \} \) is separable, and consequently, that \( X \times Z \) is separable. Let \( \{ R_1, R_2, \ldots \} \) denote a countable set dense in \( X \times Z \), and for each \( n \) let \( P_n \) denote an element of \( f^{-1}(R_n) \). The set \( K' = \text{cl}(\bigcup P_n) \) is a closed subset of \( K \) such that \( f(K') = X \times Z \), so \( K' - K \) and \( K \) is separable.

It will now be shown that \( X \times Z \) satisfies the first axiom of countability.\(^4\) Let \( P \) denote an arbitrary point of \( X \times Z \). Since \( f^{-1}(P) \) is compact and \( K - f^{-1}(P) \) is separable, it follows by an easy argument that there is a countable set \( \{ Q_1, Q_2, \ldots \} \) dense in \( K - f^{-1}(P) \) such that if \( x \in f^{-1}(P) \) and \( y \in K - f^{-1}(P) \), there is a \( Q_i \) such that \( x < Q_i \leq y \) or \( y \leq Q_i < x \). For each positive integer \( n \), let \( V_n \) denote a collection to which \( v \) belongs if and only if there is a point \( z \) of \( f^{-1}(P) \) such that \( v \) is the maximal convex subset of \( K \) which contains \( z \) and does not intersect \( \bigcup Q_i \). Since, for each \( n \), \( V_n \) is an open subset of \( K \) containing \( f^{-1}(P) \), it follows that the set \( T_n = X \times Z - f(K - V_n) \) is an open subset of \( X \times Z \) containing \( P \). Suppose \( Q \) is an arbitrary point of \( X \times Z \) distinct from \( P \), that \( z \in f^{-1}(Q) \), and also, for example, that \( z_1 \) is the last point of \( f^{-1}(P) \) which precedes \( z \) and \( z_2 \) is the first point of \( f^{-1}(P) \) which follows \( z \). There exist an integer \( j_1 \) and an integer \( j_2 \).

\(^4\) It also may be shown from \([2]\) that \( X \times Z \) satisfies the first axiom of countability.
such that \( z_1 < q_1 < z \leq q_2 < z_2 \). The set \( V_j \), where \( j = \max(j_1, j_2) \), does not contain \( z \), so the set \( T_j \) does not contain \( Q \). Therefore, \( T_1, T_2, \ldots \) is a countable sequence of open sets having only \( P \) in common.

The set \( f^{-1}(X \times \{b_1\}) \) is not metrizable, since that would imply that \( X \) is metrizable. Since Lemma 2 implies that \( f^{-1}(X \times \{b_1\}) \) is separable, Lemma 3 implies that \( G_1 = G(f^{-1}(X \times \{b_1\})) \) is uncountable. There does not exist an uncountable subcollection \( U_1 \) of \( G_1 \) such that if \((x, y) \in U_1\), then \( f(x) = f(y) \); for if there does, the conditions on \( f \) imply that for \((x, y) \in U_1\), there is a \( P \) such that \( x < P < y \), which is a contradiction. Suppose there is an uncountable subcollection \( U_2 \) of \( G_1 \) and a point \( x \) of \( X \) such that if \((z, w) \in U_2\), then \( f(z) = (x, b_1) \) or \( f(w) = (x, b_1) \). There is an uncountable subcollection \( U_3 \) of \( U_2 \) such that if \((z, w) \in U_3\) and \( f(z) = (x, b_1) \), then \( f(w) \neq (x, b_1) \). The fact that \( X \times Z \) has a countable basis at \((x, b_1)\) implies that there is an open set \( U \) containing \((x, b_1)\) and an uncountable subcollection \( U_4 \) of \( U_3 \) such that if \((z, w) \in U_4\) and \( f(z) \in U \), then \( f(w) \in (X \times Z) - U \). There is a point \( t \) of \( K \) such that each open set containing \( t \) contains uncountably many elements \((z, w)\) of \( U_4 \). The continuity of \( f \) would imply that \( f(t) = (x, b_1) \) and that \( f(t) \in (X \times Z) - U \), which is a contradiction.

Let \( C \) denote the collection of all subsets \( M \) of \( G_1 \) such that if \((p, q)\) and \((p', q')\) are elements of \( M \) then \( f(p), f(q), f(p'), \) and \( f(q') \) are four distinct points. \( C \) is partially ordered by inclusion, and each chain has an upper bound, so Zorn's lemma implies the existence of a maximal element \( W \). Suppose \( W \) is countable. Let \( D \) denote the set of all elements \((p, q)\) of \( G_1 \) such that there is an element \((p', q')\) of \( W \) such that \( f(p) = f(p') \) or \( f(q') \), or \( f(q) = f(p') \) or \( f(q') \). \( D \) is countable, so there is an element \((p, q)\) of \( G_1 - D \) such that \( f(p) \neq f(q) \). However, \( W + \{(p, q)\} \) is an element of \( C \) containing \( W \), so \( W \) is not maximal. This is a contradiction, so \( W \) is uncountable.

It will now be shown that if \( x_1 \) and \( x_2 \) are points in \( X \), then there is a positive integer \( N \) such that if \( n > N \), \( x_1 \in f^{-1}(x_1, b_n) \), and \( x_2 \in f^{-1}(x_2, b_n) \), then there is a point of \( K \) between \( x_1 \) and \( x_2 \). On the contrary, suppose there exist points \( x_1 \) and \( x_2 \) of \( X \) and an increasing sequence of integers \( n_1, n_2, \ldots \) such that for each \( i \) there exist points \( z_i \) and \( w_i \) of \( f^{-1}(x_1, b_{n_i}) \) and \( f^{-1}(x_2, b_{n_i}) \), respectively, such that no point of \( K \) lies between \( z_i \) and \( w_i \). There is a point \( z \) of \( K \) such that each open set about \( z \) contains, for infinitely many integers \( i \), both \( z_i \) and \( w_i \). But the continuity of \( f \) would imply that \( f(z) = (x_1, b) \) and also that \( f(z) = (x_2, b) \), which is a contradiction.

Let \( V \) denote the set of all ordered pairs \((x, y)\) such that there is an element \((z, w)\) of \( W \) such that \( f(z) = (x, b_1) \) and \( f(w) = (y, b_1) \). There is a positive integer \( N \) and an uncountable subcollection \( V_1 \) of \( V \) such that if \((x, y) \in V_1\), \( z \in f^{-1}(x, b_N) \), and \( w \in f^{-1}(y, b_N) \), then there
is some point of $K$ between $z$ and $w$. Let $T_1$ denote a set to which $t$ belongs if and only if there exist integers $i$ and $j$ such that $t$ is maximal with respect to the property of being a convex subset of $K$ which contains neither $P_i$ nor $P_j$. Let $T_2$ denote a collection to which $t$ belongs if and only if $t \in T_1$ or $t$ is the union of a finite number of elements of $T_1$. The collection $T_2$ is countable and has the property that if $(x, y) \in V_1$ then there exist elements $t_1$ and $t_2$ of $T_2$ such that $f^{-1}(x, b_n) \subset t_1 \subset f^{-1}(y, b_n)$ and $f^{-1}(y, b_n) \subset t_2 \subset f^{-1}(x, b_n)$. This is easily seen, because for each $z$ in $f^{-1}(x, b_n)$, for example, there is an element $t_z$ of $T_1$ which contains $z$ and does not intersect $f^{-1}(y, b_n)$, and $f^{-1}(x, b_n)$ is covered by a finite number of the $t_z$'s.

Let $S_1$ denote a collection to which an element $s$ belongs if and only if there is an element $t$ of $T_2$ such that $(s \times \{b_n\}) = X \times \{b_n\} - f(K - t) \cdot (X \times \{b_n\})$. $S_1$ is a countable collection of open subsets of $X$ such that if $(x, y) \in V_1$, there exist elements $s_1$ and $s_2$ of $S_1$ such that $x \in s_1 \subset X - \{y\}$ and $y \in s_2 \subset X - \{x\}$. Since $S_1$ is countable and $V_1$ is uncountable, there is an element $s$ of $S_1$ and an uncountable subcollection $V_2$ of $V_1$ such that if $(x, y) \in V_2$, then $x \in s \subset X - \{y\}$. Since $f$ is continuous and $s \times \{b_1\}$ is open in $X \times Z$, it follows that $f^{-1}(s \times \{b_1\})$ is open in $K$.

Let $W_1$ denote the collection of all elements $(c, d)$ of $W$ such that there is an element $(x, y)$ of $V_2$ such that $(f(c); f(d)) = (x, b_1; y, b_1)$. If $(c, d) \in W_1$, $c \in f^{-1}(s \times \{b_1\})$ and $d \in K - f^{-1}(s \times \{b_1\})$. For each pair $(c, d)$ of $W_1$ let $U(c)$ denote a convex open subset of $K$ such that $c \in U(c)$ and $U(c) \subset f^{-1}(s \times \{b_1\})$. The set of all $U(c)$'s is uncountable and no two of them intersect, so $K$ is not separable. This yields a contradiction, so $X$ is metrizable.

One interesting application of the preceding theorem is the following

**Theorem.** If a space $X$ is the continuous image of a compact ordered space and can be expressed as an infinite product $(\prod X_i)$, where each $X_i$ is a nondegenerate compact Hausdorff space, then (1) the product is a countable product, and (2) each $X_i$ is metrizable.

**References**


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