A NOTE ON COMPACT SEMIRINGS

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By a topological semiring we mean a Hausdorff space $S$ together with two continuous associative operations on $S$ such that one (called multiplication) distributes across the other (called addition). That is, we insist that $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$ for all $x$, $y$, and $z$ in $S$. Note that, in contrast to the purely algebraic situation, we do not postulate the existence of an additive identity which is a multiplicative zero.

In this note we point out a rather weak multiplicative condition under which each additive subgroup of a compact semiring is totally disconnected. We also give several corollaries and examples.

Following the notation current in topological semigroups we let $H[+]\langle e \rangle$ represent the maximal additive subgroup containing an additive idempotent $e$. Similarly $H[\cdot]\langle f \rangle$ will denote the maximal multiplicative group of a multiplicative idempotent $f$. The minimal closed additive or multiplicative semigroup containing $x$ is denoted by $\Gamma[+]\langle x \rangle$ or $\Gamma[\cdot]\langle x \rangle$ respectively. By $E[+]$ or $E[\cdot]$ we mean the collection of additive or multiplicative idempotents. Finally $A^*$ represents the topological closure of $A$. For references on the properties of these sets the reader may see [1].

**Theorem.** If $S$ is a compact semiring such that $E[\cdot]S \cup SE[\cdot] = S$ then each additive subgroup of $S$ is totally disconnected.

**Proof.** Let $S$ be a compact semiring such that $E[\cdot]S \cup SE[\cdot] = S$ and let $G$ be an additive subgroup with identity $e$. Suppose $G$ is not totally disconnected. Then neither is $H[+]\langle e \rangle$ since $G \subseteq H[+]\langle e \rangle$. That is, $C$, the component of $e$ in $H[+]\langle e \rangle$, is nontrivial. Now $C$ is a compact connected nontrivial group. It is well known [2, pp. 175, 190, and 192] that $C$ must contain a nontrivial additive one parameter group $T$. Pick $t$ different from $e$ in $T$. Recall that $t \in fS$ or $Sf$ for some $f$ in $E[\cdot]$. Suppose $t \in fS$. Clearly $fS$ is a compact subsemiring for which $f$ is a multiplicative left identity. Thus $ft=t$ so $fT$ is nontrivial and of course, $fT$ is connected. Therefore $fS$ contains a connected nontrivial group. Similarly if $t \in Sf$ then $Sf$ is a compact subsemiring with right identity containing a connected nontrivial subgroup. Thus, without loss of generality, we may assume $S$ has a left or a right identity 1.

Suppose 1 is a left identity. We identify each positive integer $n$ with...
the $n$-fold sum of 1. Thus, if $x \in S$, we may regard $nx$ as a product as well as a sum in $S$. Now, for each positive integer $n$, we have $nT = T$ so $(nT)^* = T^*$. This and the compactness of $S$ gives us $nT^* = T^*$. From a theorem of A. D. Wallace [3], we see that $xT^* = T^*$ for each $x$ in $\Gamma[+](1)$. But, $\Gamma[+](1)$ contains an additive idempotent $g$. Thus $gT^* = T^*$. On the other hand the additive idempotents $E[+]$ form a multiplicative ideal, so $T^* \subseteq E[+]$. Thus the additive group $T$ must consist of a single element. This is a contradiction. Since a similar argument applies in case $S$ contains a right identity, the theorem is proved.

By a clan we mean a compact connected semigroup with identity. Furthermore, we say a space $S$ is acyclic provided $H^n(S) = 0$ for each positive integer $n$ where $H^n(S)$ represents the $n$th Čech cohomology group of $S$.

**Corollary 1.** If $S$ is a compact semiring such that $E[\cdot]S \cup SE[\cdot] = S$ then every additive subclan is acyclic.

**Proof.** Let $T$ be an additive subclan of $S$. Let $e$ be an additive idempotent of the minimal additive ideal of $T$. Then it is known that $e+T+e$ is a group and $H^n(T) = H^n(e+T+e)$ for each positive integer $n$ [4]. Now, by the theorem, $e+T+e$ is totally disconnected. But $e+T+e$, being a continuous image of $T$, is connected. Thus $e+T+e$ is a single point and $H^n(T) = 0$ for $n > 0$.

A semigroup $S$ is said to be normal if $Sx = xS$ for each $x$ in $S$.

**Corollary 2.** Let $S$ be a compact semiring such that $SE[\cdot] \cup E[\cdot]S = S$. Let $T$ be an additive subsemigroup of $S$.

(i) If $T$ is a continuum then the minimal ideal of $T$ is idempotent.

(ii) If also $T$ is normal and $(E[+] \cap T) + T = T$ then each closed ideal of $T$ is acyclic.

(iii) If also $T$ is metric then $T$ is arcwise connected.

(iv) If also $T = S$ then there is an element $k$ in $S$ such that $k+S = S+k = k = k^2$.

**Proof.** Let $S$ be a compact semiring such that $SE[\cdot] \cup E[\cdot]S = S$ and $T$ be an additive semigroup of $S$. Suppose $T$ is a continuum. It is well known [5] that the minimal ideal of $T$ is the union of groups each of which is of the form $e+T+e$ where $e \in E[+] \cap T$. Now $e+T+e$, being a continuous image of $T$, is a continuum. But each additive subgroup of $S$ must be totally disconnected. Consequently, $e+T+e$ is a single point and we have shown that the minimal ideal of $T$ consists entirely of idempotents.

Suppose also that $T$ is normal and $(E[+] \cap T) + T = T$. The nor-
mality of $T$ implies that its minimal ideal is a group. Since the minimal ideal of $T$ is also idempotent it must be a single point, say $k$. Now Corollary 3 of [6] gives us that each closed ideal of $T$ is acyclic and part (ii) is proved.

Assume in addition that $T$ is metric and select an additive idempotent $f$ of $T$. Clearly $f+T$ is a compact connected additive subgroup of $T$. Furthermore $k \in f+T$ and, because $f+T = T+f$, $f$ is an additive identity for $f+T$. Pick $x$ in $f+T$. We have $x+(f+T) = (x+f) + T = x + T = T + x = T + (f + x) = (T + f) + x = (f + T) + x$. Thus $f + T$ is additively normal. Also, by the theorem, each additive subgroup of $f + T$ is totally disconnected. Now R. P. Hunter has shown that each such semigroup contains an arc (indeed, an $I$-semigroup) from the zero to the identity [7, Theorem 1]. Since $f + T$ is metric it is arcwise connected. But $T = \bigcup \{ f + T | f \in E[+] \cap T \}$ and $k \in f + T$ for each $f$ in $E[+] \cap T$. Therefore $T$ is arcwise connected.

Suppose $T = S$ so that $S + k = k + S = k$. Now $k$ is in either $E[+]S$ or $SE[+]$. In the former case there is a $g$ in $E[+]$ such that $gk = k$. But because $k$ is an additive zero we have $k = g + k$ and $k = k + k^2$. Thus $k = k + k^2 = (g + k)k = k^2$. In case $k \in SE[+]$ a similar argument gives us that $k = k^2$ and the corollary is proved.

Recall that if the minimal ideal $K$ of a compact semigroup $S$ is idempotent then its structure is completely known [8]. That is, $K$ must be of the form $A \times B$ where multiplication is defined by $(a, b) \cdot (a', b') = (a, b')$ for any $a$ and $a'$ in $A$ and $b$ and $b'$ in $B$.

If $S$ is a semiring, let $H[+]$ denote the union of all the additive subgroups of $S$.

**Corollary 3.** Let $S$ be an additively commutative semiring such that $SE[+] \cup E[+]S = S$. If $S$ is a metric continuum and $e \in E[+]$ then $E[+]$, $E[+] + S$, $e + S$, $H[+]$ and $e + H[+]$ are arcwise connected.

**Proof.** Clearly $E[+]$ is a closed additive subsemigroup and a multiplicative ideal. Now, since $S$ is connected and $SE[+] \cup E[+]S = S$, the multiplicative ideals of $S$ are connected and $E[+]$ is a continuum. From this it follows that $E[+] + S$ is also a continuum. Furthermore, $e + S$ being a continuous image of $S$, is a continuum. Applying the third part of Corollary 2, we have that $E[+]$, $E[+] + S$, and $e + S$ are arcwise connected. Now suppose $x \in S$ and $G$ is an additive subgroup of $S$. It follows from the distributive property that $xG$ and $Gx$ are additive subgroups of $S$. Thus $xH[+] \cup H[+]x \subset H[+]$. That is, $H[+]$ is a multiplicative ideal of $S$. It is well known [1] that $H[+]$ is a closed additive subsemigroup of $S$. Thus Corollary 2 gives us that $H[+]$ is arcwise connected.
Notice that for $E[+]$ to be a continuum it is only necessary that $S$ be a continuum and $E[\cdot]S \cup SE[\cdot] = S$. In case $E[+]$ is a single point $k$ we have $k+S=S+k=kS=Sk=k$. To see this, recall that $E[+]$ is a multiplicative ideal and hence must contain the minimal such. On the other hand, the minimal additive ideal of $S$ consists of idempotents and therefore must be $k$.

A semigroup is said to be simple if it contains no proper ideals.

**Corollary 4.** If $S$ is a compact connected additively simple semiring then each multiplicative idempotent of $S$ is an additive idempotent of $S$.

**Proof.** Let $e$ be a multiplicative idempotent of $S$. Then $eS$ is a compact connected subsemiring for which $e$ is a multiplicative left identity. Now $eS$ is additively simple since it is additively a homomorphic image of $S$. Thus the first part of Corollary 2 gives us that $eS$ is additively idempotent. But $e \in eS$ so $e$ is an additive idempotent and the corollary is proved.

**Example.** Let $A$ be the field of integers mod 3 with the discrete topology and $B$ be the interval $[0, 1]$. For $x$ and $y$ in $B$, define $x+y = xy = \min \{x, y\}$. Note that $B$ is a semiring so that $A \times B$ becomes a semiring under coordinate-wise addition and multiplication. Define the equivalence relation $\alpha$ on $A \times B$ by: $(a, j) \alpha (a', j')$ if (1) $a=a'$ and $j=j'$ or (2) $j=j'=0$. Clearly $\alpha$ is a closed congruence. Thus $(A \times B)/\alpha$ is a compact connected semiring with multiplicative identity. The maximal additive subgroups of $(A \times B)/\alpha$ are of the form $(A \times \{b\})/\alpha$ and of course totally disconnected.

On the other hand let $C$ be the circle group written additively and given the multiplication $xy = 0$ for all $x$ and $y$ in $C$. According to the theorem, $C$ can not be imbedded in a compact semiring with multiplicative identity (even if the identity is isolated).

**Question.** Regarding the proof of the third part of Corollary 2, it is easily seen that $e+T$ is not only arcwise connected but also contractable. Indeed $(e+T) \cup (f+T)$ is contractable for $e$ and $f$ in $E[+] \cap T$. The referee has pointed this out and raised the question: Is $T$ contractable?

**References**

ON SOME GEOMETRIC INEQUALITIES

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1. Let \( C \) be a closed curve of class \( C^2 \) in Euclidean \( n \)-space \( E_n \). We write the equation of \( C \) as \( x = x(s) \), \( 0 \leq s \leq L(C) \), where \( s \) denotes arc length and \( L(C) \) is the length of \( C \). Denoting differentiation with respect to \( s \) by a dot, we define the total curvature of \( C \) as

\[
K(C) = \int_C \left| \frac{\dot{x}}{\|\dot{x}\|} \right| \, ds.
\]

It is proved in [1] that if \( C \) is constrained to lie in a ball of radius \( r \), then

\[
L(C) \leq rK(C).
\]

This result is a slight sharpening of an inequality of I. Fáry [2]. The proof given in [1] depends on an integralgeometric lemma for the 2-dimensional case, together with a reduction of the \( n \)-dimensional to the 2-dimensional case by developing the curve into a plane. The proof yields no information about curves for which equality occurs in (2).

In §2 we give a simple, direct proof of (2) and characterize those curves for which equality holds. We also obtain a sharpening of an inequality of Rešetnjak [3]. A generalization to surfaces is considered in §3.

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