THE HADAMARD THEOREM FOR PERMANENTS

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1. Statement of the result. Let $A = (a_{ij})$ be an $n$-square non-negative hermitian matrix. A classical inequality of Hadamard states that

$$\det A \leq \prod_{i=1}^{n} a_{ii}$$

with equality if and only if $A$ has a zero row (and column) or $A$ is a diagonal matrix: $A = \text{diag} (a_{11}, \cdots, a_{nn})$. Several generalizations are known (e.g. [1]).

Let $\text{per} A$ denote the permanent of $A$,

$$\text{per} A = \sum_{\sigma} \prod_{i=1}^{n} a_{\sigma(i)i},$$

where the summation extends over the whole symmetric group of degree $n$. It was conjectured in [4] that in analogy with (1),

$$\text{per} A \geq \prod_{i=1}^{n} a_{ii}$$

with the conditions of equality precisely those in the Hadamard determinant theorem. L. Mirsky recently listed this conjecture among several other problems concerning the permanent function [5]. This conjecture was suggested by an inequality of I. Schur [6] (see also [4]):

$$\text{per} A \geq \det A.$$

In an unsuccessful attempt [3] to prove (3) H. Minc and the present author obtained an inequality of the form

$$\text{per} A \geq c_n \prod_{i=1}^{n} a_{ii}$$

in which the constant $c_n$ depends only on $n$ and not on $A$.

It is the purpose of this paper to present the proof of an inequality that substantially generalizes (3) and to discuss the somewhat delicate cases of equality. Let $A(i)$ denote the principal submatrix of $A$ obtained by deleting row and column $i$ of $A$. The main result is contained in the following

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967
Theorem 1. Let $A = (a_{ij})$ be an $(r+1)$-square non-negative hermitian matrix. Then

$$ (4) \quad (r+1)a_{ii} \per A(i) \geq \per A \geq a_{ii} \per A(i), \quad 1 \leq i \leq n. $$

If $A$ has a zero row then (4) is equality throughout. If $A$ has no zero row then the lower equality holds if and only if $a_{ii}$ is the only nonzero entry in row and column $i$ of $A$; the upper equality holds if and only if the rank of $A$ is 1.

The permanent is unaltered by pre- and post-multiplication by permutation matrices so that we can take $i=1$ in proving (4).

Once (4) is established it is clear that an obvious induction on $r$ will yield

Theorem 2. If $A = (a_{ij})$ is an $n$-square non-negative hermitian matrix then

$$ (5) \quad \per A \geq \prod_{i=1}^{n} a_{ii} $$

with equality if and only if $A$ has a zero row or $A$ is a diagonal matrix.

2. Preliminaries. Let $U$ be an $n$-dimensional unitary space with inner product $(x, y)$. For $1 \leq r \leq n$ define $U^{(r)}$ to be the space of $r$-tensors on $U$ [2, Chapter 7]; that is, $U^{(r)}$ is the dual space of the space $M_r(U)$ of all multilinear functionals of $r$-tuples of vectors from $U$. If $x_1, \cdots, x_r$ are in $U$ then their tensor product $x_1 \otimes \cdots \otimes x_r \in U^{(r)}$ is defined by

$$ x_1 \otimes \cdots \otimes x_r(\phi) = \phi(x_1, \cdots, x_r), \quad \phi \in M_r(U). $$

An inner product in $U^{(r)}$ is given by

$$ (x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_r) = \prod_{i=1}^{r} (x_i, y_i). $$

Define the completely symmetric operator $S^{(r)}: U^{(r)} \rightarrow U^{(r)}$ by

$$ (7) \quad S^{(r)}(x_1 \otimes \cdots \otimes x_r) = \frac{1}{r!} \sum_{\sigma \in S_r} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}, $$

where $S_r$ is the symmetric group of degree $r$. The symmetric product of $x_1, \cdots, x_r$ is then defined by

$$ (8) \quad x_1 \cdots x_r = S^{(r)}(x_1 \otimes \cdots \otimes x_r). $$

The range space of $S^{(r)}$ is the symmetry class of completely symmetric.
tensors on $U$ denoted by $U_{(r)}$ and since it is a subspace of $U^{(r)}$ it is unitary. By combining (6), (7), (8) we compute that the inner product of two symmetric products is

$$ (x_1 \cdots x_r, y_1 \cdots y_r)_r = \frac{1}{r!} \per((x_i, y_j)), \quad 1 \leq i, j \leq r. \tag{9} $$

(The formula (9) is an immediate consequence of the fact that $S^{(r)}$ is idempotent and hermitian.) Next let $G_{r,n}$ denote the totality of non-decreasing sequences of length $r$ chosen from $1, \ldots, n$. If $\omega \in G_{r,n}$ let $\mu(\omega)$ be the product of the factorials of the multiplicities of the distinct integers in $\omega$; e.g. $\mu(3, 3, 7, 7, 7, 9) = 2!3!$. If $e_1, \ldots, e_n$ is an orthonormal (o.n.) basis of $U$ then the symmetric products $\sqrt{(r! / \mu(\omega))} e_{a_1} \cdots e_{a_r}$, $(\omega_1, \ldots, \omega_r) = \omega \in G_{r,n}$, constitute an o.n. basis of $U_{(r)}$ in the inner product $(\cdot, \cdot)_r$. We let $e_a = e_{a_1} \cdots e_{a_r}$.

3. Proofs. Assume that $A$ is $(r+1)$-square and has no zero row and let $D = \text{diag}(a_{11}^{1/2}, 1, \ldots, 1)$. Then the $1, 1$ entry of $B = DAD$ is $1$, $\per B = a_{11} \per A$, $A(1) = B(1)$, and $B$ is also non-negative hermitian. If we prove Theorem 1 for $B$ we will clearly have the result we want for $A$. Since $B$ is non-negative hermitian it follows that $B$ is a Gram matrix based on some set $e_1, v_1, \ldots, v_r$ where $e_1$ is a unit vector:

$$ b_{11} = 1 = (e_1, e_1), $$

$$ b_{1, j+1} = b_{j+1, 1} = (e_1, v_j), \quad j = 1, \ldots, r, $$

$$ b_{i+1, j+1} = (v_i, v_j), \quad i, j = 1, \ldots, r. $$

Let $e_2, \ldots, e_n$ be a completion of $e_1$ to an o.n. basis of $U$. Define a map $T: U_{(r)} \rightarrow U_{(r+1)}$ by $T(e_a) = e_1 \cdot e_{a_1} \cdots e_{a_r}$, all $\omega = (\omega_1, \ldots, \omega_r) \in G_{r,n}$, and extend linearly. It is an easy consequence of the symmetry and linearity of the symmetric product in its factors that

$$ T(x_1 \cdots x_r) = e_1 \cdot x_1 \cdots x_r $$

for all $x_1, \ldots, x_r$ in $U$. Let $R \subseteq U_{(r+1)}$ denote the range of $T$ and let $T^*$ denote the conjugate dual map of $T$; $T^*: R \rightarrow U_{(r)}$. That is, $T^*$ satisfies

$$ (Th, g)_{r+1} = (h, T^*g), \quad \text{for all } h \in U_{(r)}, g \in R. $$
Next let $f(x_1, \ldots, x_r)$ denote the Rayleigh quotient

$$f(x_1, \ldots, x_r) = \frac{(T x_1 \cdots x_r, T x_1 \cdots x_r)_{r+1}}{(x_1 \cdots x_r, x_1 \cdots x_r)_{r}}$$

(10)

$$= \frac{(T^* T x_1 \cdots x_r, x_1 \cdots x_r)_{r}}{(x_1 \cdots x_r, x_1 \cdots x_r)_{r}}.$$ 

Now let $H$ denote the non-negative hermitian transformation $T^* T : U(r) \rightarrow U(\omega)$. It is easy to verify [4, Theorem 3] that $x_1 \cdots x_r = 0$ if and only if some $x_i = 0$. Thus $f(x_1, \ldots, x_r)$ is defined for all sets of $r$ nonzero vectors $x_1, \ldots, x_r$ in $U$ and it is known that such values of $f$ lie in the interval between the largest and smallest eigenvalues of $H$. We compute the eigenvalues of $H$ by obtaining a matrix representation of $H$. The basis $\sqrt{r!(\mu(\omega))}e_\omega, \omega \in G_{r,n}$, ordered lexicographically in $\omega$, is o.n. for $U(\omega)$ and the $(\tau, \omega)$ entry of the matrix representation of $H$ in this ordered basis is

$$r! \sqrt{\mu(\omega) \mu(\tau)} (He_\omega, e_\tau)_r.$$ 

(11)

Now

$$(He_\omega, e_\tau)_r = (T^* T e_\omega, e_\tau)_r = (T e_\omega, T e_\tau)_{r+1}$$

(12)

$$= (e_1 \cdot e_{u_1} \cdots e_{u_r}, e_1 \cdot e_{r_1} \cdots e_{r_r})$$

$$= (e(1, \omega), e(1, \tau))_{r+1}$$

where $(1, \omega)$ is the sequence $(1, \omega_1, \ldots, \omega_\tau) \in G_{r+1,n}$ and similarly for $(1, \tau)$. The vectors $\sqrt{(r+1)!/\mu(\omega))}e_\alpha, \alpha \in G_{r+1,n}$, are an o.n. basis for $U_{r+1}$ in the inner product $(\ , \ )_{r+1}$. Moreover $(1, \omega) = (1, \tau)$ if and only if $\omega = \tau$. Hence from (12) we have

$$(He_\omega, e_\tau)_r = \frac{\mu((1, \omega))}{r+1} \delta_{u_\tau}.$$ 

Thus the matrix representation of $H$ in the basis $\sqrt{r!/\mu(\omega)}e_\omega$ is diagonal and the eigenvalues of $H$ are seen from (11) to be

$$\lambda_\omega(H) = \frac{r!}{\mu(\omega)} (He_\omega, e_\omega)_r$$

$$= \frac{r!}{\mu(\omega)} \frac{\mu((1, \omega))}{(r+1)!}$$

$$= \frac{1}{r+1} \frac{\mu((1, \omega))}{\mu(\omega)}.$$
Clearly \( \mu((1, \omega)) \geq \mu(\omega) \) with equality if and only if \( \omega_1 > 1 \). Thus the minimum eigenvalue of \( H \) is \( 1/(r+1) \) and the symmetric tensors \( e_\omega, \omega_1 > 1 \), constitute the totality of corresponding eigenvectors. Suppose the multiplicities greater than 1 of the distinct integers in \( \omega \) are \( m_1, \cdots, m_p \). Then the multiplicities for \( (1, \omega) \) are either \( m_1, \cdots, m_p \) or \( m_1+1, m_2, \cdots, m_p \). In the first instance \( \omega_1 > 1 \) and \( \mu((1, \omega))/\mu(\omega) = 1 \); in the second \( \omega_1 = 1 \) and \( \mu((1, \omega))/\mu(\omega) = m_1+1 \). This latter expression is maximal only when \( m_1 = r \), i.e., for the sequence \( \omega = (1, \cdots, 1) \). Thus we conclude that

\[
1 \geq f(x_1, \cdots, x_r) \geq \frac{1}{r+1}.
\]

The lower equality holds if and only if \( x_1 \cdots x_r \) lies in the space spanned by the tensors \( e_\omega, \omega_1 > 1, \omega \in G_{r,n} \). The upper equality holds if and only if \( x_1 \cdots x_r \) is a multiple of \( e_1 \cdots e_1 \). Now by (10) we have

\[
f(v_1, \cdots, v_r) = \frac{(T_{v_1} \cdots v_r, T_{v_1} \cdots v_r)_{r+1}}{(v_1 \cdots v_r, v_1 \cdots v_r)_r} = \frac{(e_1 \cdot v_1 \cdots v_r, e_1 \cdot v_1 \cdots v_r)_{r+1}}{(v_1 \cdots v_r, v_1 \cdots v_r)_r} = \frac{1}{(r+1)!} \frac{1}{\text{per } B}
\]

and it follows from (13) that

\[
(r+1) \text{ per } B(1) \geq \text{ per } B \geq \text{ per } B(1).
\]

As we indicated earlier (4) follows from (14).

We noted following (13) that the lower equality can hold if and only if

\[
v_1 \cdots v_r = \sum c_\omega e_\omega
\]

where the summation extends over all \( \omega \in G_{r,n} \) for which \( \omega_1 > 1 \). We prove that (15) implies that \( \langle v_i, e_i \rangle = 0, i = 1, \cdots, r \). Let \( h \) denote the tensor on the right side of (15). Then
\[(v_1 \cdots v_r, e_1 \cdots e_1)_r = \prod_{k=1}^{r} (v_k, e_1)\]

and

\[(h, e_1 \cdots e_1)_r = \sum_{a_1>1} c_a (e_a, e_1 \cdots e_1)_r = 0.\]

Hence some \((v_j, e_1) = 0\) and from the symmetry of the symmetric product we can assume \(j = 1\). Suppose we have proved that \((v_1, e_1) = \cdots = (v_k, e_1) = 0\). Since no \(v_i = 0, \ i = 1, \cdots, r\), we know [4, Theorem 3] that \(0 \neq v_1 \cdots v_k \in U(k)\). The tensors \(e_\alpha, \alpha \in G_{k,n}\), constitute a basis for \(U(k)\) and because \(v_1 \cdots v_k \neq 0\) there exists an \(\alpha \in G_{k,n}\) for which \((v_1 \cdots v_k, e_\alpha)_k \neq 0\). Let \(\beta = (1, \cdots, 1, \alpha_1, \cdots, \alpha_k) \in G_{r,n}\) where the notation means that \(\alpha\) has been preceded with \(r-k\) 1's to make a nondecreasing sequence of length \(r\). Now once again \((h, e_\beta) = 0\) and hence

\[0 = (v_1 \cdots v_r, e_\beta)_r = (v_1 \cdots v_k \cdot v_k \cdot v_{k+1} \cdots v_r, e_1 \cdots e_1 \cdot e_{a_1} \cdots e_{a_k})_r\]

\[= \frac{1}{r!} \text{per} \begin{bmatrix}
(v_1, e_1) & \cdots & (v_1, e_1) & (v_1, e_{a_1}) & \cdots & (v_1, e_{a_k}) \\
(v_k, e_1) & \cdots & (v_k, e_1) & (v_k, e_{a_1}) & \cdots & (v_k, e_{a_k}) \\
(v_{k+1}, e_1) & \cdots & (v_{k+1}, e_1) & (v_{k+1}, e_{a_1}) & \cdots & (v_{k+1}, e_{a_k}) \\
\vdots & & \vdots & & \vdots & \\
(v_r, e_1) & \cdots & (v_r, e_1) & (v_r, e_{a_1}) & \cdots & (v_r, e_{a_k})
\end{bmatrix}.
\]

The upper left block in this last matrix is \(k \times (r-k)\), the upper right block is \(k \times k\), the lower left is \((r-k) \times (r-k)\) and the lower right is \((r-k) \times k\). The upper left block consists of zeros and hence by the Laplace expansion theorem for permanents we have

\[0 = \text{per} \begin{bmatrix}
(v_1, e_{a_1}) & \cdots & (v_1, e_{a_k}) \\
\vdots & & \vdots \\
(v_k, e_{a_1}) & \cdots & (v_k, e_{a_k})
\end{bmatrix} \text{ per} \begin{bmatrix}
(v_{k+1}, e_1) & \cdots & (v_{k+1}, e_1) \\
(v_r, e_1) & \cdots & (v_r, e_1)
\end{bmatrix}
\]

\[= k! (v_1 \cdots v_k, e_{a_1}) (r-k)! \prod_{j=k+1}^{r} (v_j, e_1).
\]

Now \((v_1 \cdots v_k, e_{a_1}) \neq 0\) and hence some \((v_j, e_1) = 0, \ j = k+1, \cdots, r\). We can assume \((v_{i+1}, e_1) = 0, \ i = 1, \cdots, r\). In terms of the matrix \(B\) this implies that \(b_{ij} = b_{ji} = 0, \ j = 2, \cdots, r+1\), and hence \(B = 1 + B(1)\).
If the upper equality holds then $v_1 \cdots v_r$ was seen to be a multiple of $e_1 \cdots e_1$. Since we are assuming that $A$ has no zero row it follows from [4, Theorem 3] that $v_i = d_i e_i$, $i = 1, \ldots, r$, and hence $B$ is the Gram matrix based on the set $e_1, d_1 e_1, \ldots, d_r e_1$. This means that $B$ and hence $A$ has rank 1, say $A = (z_i z_j)$, $i, j = 1, \ldots, r+1$. Of course, when $A$ has this form then
\[
\operatorname{per} A = (r+1) ! \prod_{j=1}^{r+1} |z_j|^2, \quad \text{per } A(1) = r ! \prod_{j=2}^{r+1} |z_j|^2 \]
and clearly $\operatorname{per} A = (r+1)a_{11}$ per $A(1)$. This completes the proof of Theorem 1.

References


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