SHORTER NOTES

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THE MARCINKIEWICZ INTERPOLATION THEOREM

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We show that the Marcinkiewicz Theorem on the interpolation of operators acting on $L^p$ spaces (see [3, pp. 111-116]) is an immediate consequence of two easily proved inequalities. The first one is a well-known result of Hardy (see [1, pp. 245-246]):

If $q \geq 1$, $r > 0$, and $g$ is a measurable, non-negative function on $(0, \infty)$, then

$$\left( \int_0^\infty \left( \int_0^t g(y) dy \right)^q t^{-r+1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty (yg(y))^{qy-r+1} dy \right)^{1/q}$$

(1)

$$\left( \int_0^\infty \left( \int_t^\infty g(y) dy \right)^q t^{-r+1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty (yg(y))^{qy-r+1} dy \right)^{1/q}.$$

The second one can be found in [2]:

If $g$ is non-negative and nonincreasing on $(0, \infty)$, $1 \leq q_1 \leq q_2 \leq \infty$ and $1 \leq p \leq \infty$, then

$$\left( \int_0^\infty \left[ t^{1/q_2} g(t) \right]^{q_2} dt \right)^{1/q_2} \leq \left( \frac{q_1}{p} \right)^{1/q_1-1/q_2} \left( \int_0^\infty \left[ t^{1/q_1} g(t) \right]^{q_1} dt \right)^{1/q_1}.$$

(2)

If $h$ is measurable on a measure space $M$ with measure $m$, its distribution function is defined for $y > 0$ by $\lambda_h(y) = m \{ x \in M; f(x) > y \}$.

The nonincreasing rearrangement of $h$ onto $(0, \infty)$ is then the function given by $h^*(t) = \inf \{ y > 0; \lambda(h) \leq t \}$, $t > 0$. Both $h^*$ and $\lambda$ are non-negative and nonincreasing functions that are continuous from the right. $h^*$ and $h$ have the same distribution function, thus $\|h^*\|_p = \|h\|_p$. Moreover, $\sup_{t > 0} \lambda(h^*)(y) \leq \sup_{t > 0} t^{1/q} h^*(t)$. Consequently the theorem of Marcinkiewicz can be stated in the following way:

Suppose $T$ is quasi-linear and, for $1 \leq p_i \leq q_i \leq \infty$, $i = 0, 1$, with $p_0 < p_1$, $q_0 \not= q_1$,

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2 An operator $T$ mapping functions on a measure space into functions on another measure space is called quasi-linear if $T(f+g)$ is defined whenever $Tf$ and $Tg$ are defined and if $||T(f+g)(x)|| \leq K(||Tf(x)|| + ||Tg(x)||)$ a.e., where $K$ is a positive constant independent of $f$ and $g$. 996
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(3) \[ \sup_{t>0} t^{(1/p_i)}h^*(t) \leq B_i \| f \|_{p_i} \quad \text{for all } f \text{ in } L^{p_i}, \quad i = 0, 1, \]

where \( h = Tf \) and \( B_0, B_1 \) are independent of \( f \). Then, for \( 0 < \theta < 1 \), there exists \( B = B_\theta \) such that \( \| h \|_q = \| Tf \|_q \leq B \| f \|_p \) for all \( f \) in \( L^p \), \( 1/p = (1-\theta)/p_0 + \theta/p_1 \) and \( 1/q = (1-\theta)/q_0 + \theta/q_1 \).

**Proof.** Put

\[ f^t(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(\tau), \\ 0 & \text{otherwise}, \end{cases} \]

and \( f_t = f - f^t \), where

\[ \gamma = \frac{(1/q_0) - (1/q)}{(1/p_0) - (1/p)} = \frac{(1/q) - (1/q_1)}{(1/p) - (1/p_1)}. \]

Then it follows easily that

\[ f^t(y) \leq \begin{cases} f^*(y) & \text{if } 0 < y < \tau, \\ 0 & \text{if } y \geq \tau, \end{cases} \]

(4) \[ f^*(y) \leq \begin{cases} f^*(\tau) & \text{if } 0 < y < \tau, \\ f^*(y) & \text{if } y \geq \tau. \end{cases} \]

Suppose \( p_1 < \infty \). Using (2),

\[ \| Tf \|_q = \left( \int_0^\infty [t^{(1/p)}(Tf)^*(t)]^q \frac{dt}{t} \right)^{1/q} \]

\[ \leq \left( \frac{p}{q} \right)^{(1/p) - (1/q)} \left( \int_0^\infty [t^{(1/p)}(Tf)^*(t)]^{p/q} \frac{dt}{t} \right)^{1/p}. \]

It follows easily from the definitions that

\[ (T(f_i + f^t))^*(t) \leq 2K \left( (Tf_i)^* \left( \frac{t}{2} \right) + (Tf^t)^* \left( \frac{t}{2} \right) \right). \]

Using this, a change of variables and Minkowski’s inequality we majorize the above by

\[ (2K)2^{1/q} \left( \frac{p}{q} \right)^{1/p - 1/q} \left\{ \left( \int_0^\infty [t^{1/q}(Tf_i)^*(t)]^p \frac{dt}{t} \right)^{1/p} + \left( \int_0^\infty [t^{1/q}(Tf^t)^*(t)]^p \frac{dt}{t} \right)^{1/p} \right\} \]

By (3), this is dominated by
which, by (4), (2) and Minkowski’s inequality is majorized by

\[(2K)^{2/q} \left( \frac{p}{q} \right)^{1/p - 1/q} \left\{ \left( \int_0^\infty \left[ t^{1/q - 1/q_2} \left( \int_0^t y^{1/p_0} f^*(y) \frac{dy}{y} \right) \right]^{p_0} \frac{dt}{t} \right)^{1/p_0} 
+ \left( \int_0^\infty \left[ t^{1/q - 1/q_1} \left( \int_0^t y^{1/p_1} f^*(y) \frac{dy}{y} \right) \right]^{p_1} \frac{dt}{t} \right)^{1/p_1} \right\}, \]

Finally, by a change of variables and (1) the last expression is less than or equal to

\[(2K)^{2/q} \left( \frac{p}{q} \right)^{1/p - 1/q} \left| \gamma \right|^{-1/p} \left\{ B_0 \left( \frac{1}{p_0} \right)^{1-1/p_0} + B_1 \left( \frac{1}{p_1} \right)^{1-1/p_1} \right\} \left\| f \right\|_p = B\left\| f \right\|_p. \]

In case \( p_1 = q_2 = \infty \) the proof is the same except for the use of the estimate \( \left\| f \right\|_\infty = \left\| f^* \right\|_1 \).

### Bibliography