A NOTE ON VARIATIONAL METHODS

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In this note a new development of the variational method due to G. M. Golusin will be given. The Golusin variational method, found in Geometrische Funktionentheorie [1, pp. 96–105], is established there only after rather lengthy and tedious considerations. Below, the interior variational formula of M. M. Schiffer [2] is used and the Golusin variation is quickly and easily obtained.

The Schiffer variation as used herein may be stated as follows. If

\[ z^* = z + \varepsilon k(z) \]

(\( \varepsilon \) small) maps the boundary \( C \) of the simply-connected domain \( D \) schlicht onto the boundary \( C^* \) of the simply-connected domain \( D^* \), then the variation \( \delta g(\zeta, \eta) = g^*(\zeta, \eta) - g(\zeta, \eta) \) of the Green's function is given by the formula

\[ \delta g(\zeta, \eta) = \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_C p'(z, \eta) p'(z, \zeta) \varepsilon k(z) dz \right\} + O(\varepsilon^2), \]

where \( p(z, \eta) \) is the analytic completion of the Green's function \( g(z, \eta) \) of \( D \) and \( \Gamma \) is a curve in \( D \) homotopic to \( C \) and such that the domain \( \Delta \) bounded by \( \Gamma \) is contained in \( D^* \).

Theorem (Golusin). Let \( z = f(w) \), regular and schlicht in \( |w| < 1 \), map \( |w| < 1 \) onto a domain \( D \) containing \( z = 0 \) in such a way that \( f(0) = 0 \). Let \( z^* = F(w, \varepsilon) \) be regular as a function of \( w \) and \( \varepsilon \) for \( r \leq |w| < 1 \) and \( |\varepsilon| < \varepsilon_0 \); and for each \( \varepsilon > 0 \), \( 0 < \varepsilon < \varepsilon_0 \), let \( z^* = F(w, \varepsilon) \) be schlicht in \( r \leq |w| < 1 \). Furthermore, suppose that for all \( w \) in \( r \leq |w| < 1 \) and small \( \varepsilon \) we have

\[ F(w, \varepsilon) = f(w) + \varepsilon q(w) + O(\varepsilon^2). \]

Denote by \( D^* \) the simply-connected domain obtained by taking the union of the image of \( r \leq |w| < 1 \) (under \( z^* = F(w, \varepsilon) \)) and the domain bounded by the image of \( |w| = r \) (under \( z^* = F(w, \varepsilon) \)). Then for one of the func-

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tions \( z^* = f^*(w) \), \( f^*(0) = 0 \), which maps \(|w| < 1 \) schlicht onto \( D^* \), we have in \(|w| < 1 \):

\[
(3) \quad f^*(w) = f(w) + \epsilon q(w) - \epsilon wf'(w)S(w) + \epsilon wf'(w)S\left(\frac{1}{w}\right) + O(\epsilon^2),
\]

where \( S(w) \) is the sum of terms involving negative powers of \( w \) in the Laurent expansion of \( q(w)/wf'(w) \) in the ring \( r < |w| < 1 \).

**Proof.** Let \( w = \phi(z) \) be the inverse of \( z = f(w) \). Let \( \omega \in \{w: |w| < 1\} \) be arbitrary and let \( \xi = f(\omega) \).

If in (1) we let \( k(z) = q(\phi(z)) + O(\epsilon) \), then

\[
z^* = z + \epsilon k(z) = z + \epsilon q(\phi(z)) + O(\epsilon^2)
\]

varies the domain \( D \) to give the domain \( D^* \). Schiffer’s variational formula (2), with \( \eta = 0 \), thus becomes

\[
(4) \quad \delta g(\xi, 0) = \text{Re} \left\{ \frac{1}{2\pi i} \int \frac{p'(z, 0) p'(z, \xi) q(\phi(z)) dz}{w} \right\} + O(\epsilon^2).
\]

Now

\[
p(z, \xi) = - \log \frac{\phi(z) - \phi(\xi)}{1 - [\phi(\xi)]^{-1} \phi(z)}
\]

and

\[
p'(z, \xi) = - \frac{[\phi(\xi)]^{-1} \phi'(z)}{1 - [\phi(\xi)]^{-1} \phi(z)} - \frac{\phi'(z)}{\phi(z) - \phi(\xi)}.
\]

In view of this and the fact that \( \phi'(z) = 1/f'(w) \) we have

\[
\frac{1}{2\pi i} \int \frac{p'(z, 0) p'(z, \xi) q(\phi(z)) dz}{w} = \frac{q(w)}{w f'(w)} \frac{1}{w - \omega} - \frac{1}{f'(w)(w - 1/\omega)} \int q(w) f'(w) dw
\]

which we wish to evaluate.

Let \( \sum_{n=-\infty} a_n w^n \) be the Laurent expansion of \( q(w)/wf'(w) \) about \( w = 0 \). Now
\[
\frac{1}{w - \omega} = \frac{1}{w} \frac{1}{1 - \omega / w} = \sum_{n=0}^{\infty} \omega^n w^{n-1}
\]

and
\[
\frac{1}{w - 1/\omega} = -\frac{1}{\omega} \frac{1}{1 - \omega / w} = -\sum_{k=0}^{\infty} \omega^{k+1} w^k.
\]

Thus
\[
\frac{q(w)}{wf'(w)} \frac{1}{w - \omega} = \sum_{n=-\infty}^{\infty} a_n \omega^n \sum_{n=0}^{\infty} \omega^n w^{n-1}
\]
\[
= \sum_{\mu=-\infty}^{\infty} \sum_{n=\mu+1}^{\infty} a_n \omega^{n-\mu} w^\mu.
\]  
(5)

and
\[
\frac{q(w)}{wf'(w)} \frac{1}{w - 1/\omega} = -\sum_{n=-\infty}^{\infty} a_n \omega^n \sum_{k=0}^{\infty} \omega^{k+1} w^k
\]
\[
= -\sum_{\mu=-\infty}^{\infty} \sum_{n=\mu+1}^{\infty} a_n \omega^{n-\mu} w^\mu.
\]  
(6)

The coefficient of \(w^{-1}\) is \(\sum_{n=0}^{\infty} a_n \omega^n\) in (5) and \(-\sum_{n=-\infty}^{-1} a_n \omega^n\) in (6). Thus, by the residue theorem, we have
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{y'(z, 0) y'(z, \xi) eq(\phi(z))}{\omega f'(\omega)} dz
\]
\[
= \varepsilon \sum_{n=0}^{\infty} a_n \omega^n + \varepsilon^{-1} \sum_{n=-\infty}^{n=0} a_n \omega^{-n}
\]
\[
= \varepsilon \sum_{n=0}^{\infty} a_n \omega^n - \varepsilon^{-1} \sum_{n=-\infty}^{n=0} a_n \omega^n + \varepsilon^{-1} \sum_{n=-\infty}^{n=0} a_n (1/\omega)^n
\]
\[
= \frac{eq(\omega)}{\omega f'(\omega)} - \varepsilon S(\omega) + \varepsilon S(1/\omega).
\)

And with this, equation (4) becomes
\[
\delta_g(z, 0) = \text{Re} \left\{ \frac{eq(\omega)}{\omega f'(\omega)} - \varepsilon S(\omega) + \varepsilon S(1/\omega) \right\} + O(\varepsilon^2),
\]

where we conjugate \(S(1/\omega)\) so that the expression in braces is analytic.
Let $\phi^*_1(\zeta)$ map $D^*$ schlichtly onto $|w| < 1$ such that $\phi^*_1(0) = 0$. Now $g(\zeta, 0) = -\log |\phi(\zeta)|$ so
\[
\log |\phi^*_1(\zeta)| - \log |\phi(\zeta)| = \delta g(\zeta, 0) = \Re\{K\} + O(\epsilon^2),
\]
where $K = \epsilon g(\omega)/\omega f'(\omega) - \epsilon S(\omega) + \epsilon S(1/\omega)$. Completing this analytically and taking exponentials we have
\[
\phi^*_1(\zeta)e^{-i\epsilon} - \phi(\zeta) = -\phi(\zeta)K + O(\epsilon^2),
\]
where $\epsilon$ is a real constant. Let $\phi^*(\zeta) = \phi^*_1(\zeta)e^{-i\epsilon}$ and let $f^*$ be the inverse of $\phi^*$. Then
\[
\phi^*(\zeta) - \phi(\zeta) = -\phi(\zeta)K + O(\epsilon^2)
\]
and since $f^*(\omega^*) = f^*(\phi^*(\zeta)) = \zeta = f(\omega),$
\[
f(\omega) = f^*(\omega^*) = f^*(\omega - \omega K + O(\epsilon^2))
= f^*(\omega) - f^*(\omega)\omega K + O(\epsilon^2)
\]
\[
= f^*(\omega) - f'(\omega)\omega K + O(\epsilon^2),
\]
since $f^*(\omega) = f'(\omega) + O(\epsilon^2)$. This completes the proof since $\omega$ was arbitrary in $|w| < 1$, i.e.,
\[
f^*(w) = f(w) + \epsilon g(w) - \epsilon w f'(w)S(w) + \epsilon w f'(w)\overline{S(1/\omega)} + O(\epsilon^2)
\]
is valid in $|w| < 1$.

References


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