A NOTE ON VARIATIONAL METHODS

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In this note a new development of the variational method due to G. M. Golusin will be given. The Golusin variational method, found in *Geometrische Funktionentheorie* [1, pp. 96–105], is established there only after rather lengthy and tedious considerations. Below, the interior variational formula of M. M. Schiffer [2] is used and the Golusin variation is quickly and easily obtained.

The Schiffer variation as used herein may be stated as follows. If

$$z^* = z + \epsilon k(z) \tag{1}$$

(ϵ small) maps the boundary C of the simply-connected domain D schlicht onto the boundary C^* of the simply-connected domain D^*, then the variation $\delta g(\zeta, \eta) = g^*(\zeta, \eta) - g(\zeta, \eta)$ of the Green's function is given by the formula

$$\delta g(\zeta, \eta) = \operatorname{Re}\left\{\frac{1}{2\pi i}\int_\Gamma p'(z, \eta)p'(z, \zeta)\epsilon k(z)dz\right\} + O(\epsilon^2), \tag{2}$$

where $p(z, \eta)$ is the analytic completion of the Green's function $g(z, \eta)$ of D and Γ is a curve in D homotopic to C and such that the domain Δ bounded by Γ is contained in D^*.

THEOREM (GOLUSIN). *Let $z=f(w)$, regular and schlicht in $|w|<1$, map $|w|<1$ onto a domain D containing $z=0$ in such a way that $f(0)=0$. Let $z^*=F(w, \epsilon)$ be regular as a function of w and ϵ for $r \leqq |w| < 1$ and $|\epsilon| < \epsilon_0$; and for each ϵ, $0<\epsilon<\epsilon_0$, let $z^*=F(w, \epsilon)$ be schlicht in $r \leqq |w| < 1$. Furthermore, suppose that for all w in $r \leqq |w| < 1$ and small ϵ we have*

$$F(w, \epsilon) = f(w) + \epsilon q(w) + O(\epsilon^2).$$

Denote by D^* the simply-connected domain obtained by taking the union of the image of $r \leqq |w| < 1$ (under $z^*=F(w, \epsilon)$) and the domain bounded by the image of $|w|=r$ (under $z^*=F(w, \epsilon)$). Then for one of the func-

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tions $z^*=f^*(w)$, $f^*(0)=0$, *which maps* $|w|<1$ *schlicht onto* D^*, *we have in* $|w|<1$:

$$(3)\qquad f^*(w)=f(w)+\epsilon q(w)-\epsilon wf'(w)S(w)+\epsilon wf'(w)\bar{S}\left(\frac{1}{\bar{w}}\right)+O(\epsilon^2),$$

where $S(w)$ *is the sum of terms involving negative powers of* w *in the Laurent expansion of* $q(w)/wf'(w)$ *in the ring* $r<|w|<1$.

PROOF. Let $w=\phi(z)$ be the inverse of $z=f(w)$. Let $\omega\in\{w: |w|<1\}$ be arbitrary and let $\zeta=f(\omega)$.

If in (1) we let $k(z)=q(\phi(z))+O(\epsilon)$, then

$$\begin{aligned} z^* &= z+\epsilon k(z)\\ &= z+\epsilon q(\phi(z))+O(\epsilon^2)\end{aligned}$$

varies the domain D to give the domain D^*. Schiffer's variational formula (2), with $\eta=0$, thus becomes

$$(4)\qquad \delta g(\zeta,0)=\operatorname{Re}\left\{\frac{1}{2\pi i}\int_\Gamma p'(z,0)p'(z,\zeta)\epsilon q(\phi(z))dz\right\}+O(\epsilon^2).$$

Now

$$p(z,\zeta)=-\log\frac{\phi(z)-\phi(\zeta)}{1-[\phi(\zeta)]^-\phi(z)}$$

and

$$p'(z,\zeta)=-\frac{[\phi(\zeta)]^-\phi'(z)}{1-[\phi'(\zeta)]^-\phi(z)}-\frac{\phi'(z)}{\phi(z)-\phi(\zeta)}.$$

In view of this and the fact that $\phi'(z)=1/f'(w)$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_\Gamma p'(z,0)p'(z,\zeta)\epsilon q(\phi(z))dz\\ &=\frac{1}{2\pi i}\int_{|w|=r}\frac{1}{wf'(w)}\left[\frac{1}{f'(w)(w-\omega)}-\frac{1}{f'(w)(w-1/\bar{\omega})}\right]\epsilon q(w)f'(w)dw\\ &=\frac{\epsilon}{2\pi i}\int_{|w|=r}\frac{q(w)}{wf'(w)}\frac{1}{w-\omega}dw-\frac{\epsilon}{2\pi i}\int_{|w|=r}\frac{q(w)}{wf'(w)}\frac{1}{1-1/\bar{\omega}}dw,\end{aligned}$$

which we wish to evaluate.

Let $\sum_{n=-\infty}^{\infty}a_nw^n$ be the Laurent expansion of $q(w)/wf'(w)$ about $w=0$. Now

$$\frac{1}{w-\omega} = \frac{1}{w}\,\frac{1}{1-\omega/w} = \sum_{v=0}^{\infty} \omega^v w^{-v-1}$$

and

$$\frac{1}{w-1/\bar{\omega}} = -\,\bar{\omega}\,\frac{1}{1-w\bar{\omega}} = -\sum_{k=0}^{\infty} \bar{\omega}^{k+1} w^k.$$

Thus

$$\begin{aligned} \frac{q(w)}{wf'(w)}\,\frac{1}{w-\omega} &= \sum_{n=-\infty}^{\infty} a_n w^n \sum_{v=0}^{\infty} \omega^v w^{-v-1} \\ &= \sum_{\mu=-\infty}^{\infty} \sum_{n=\mu+1}^{\infty} a_n \omega^{n-1-\mu} w^\mu \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{q(w)}{wf'(w)}\,\frac{1}{w-1/\bar{\omega}} &= -\sum_{n=-\infty}^{\infty} a_n w^n \sum_{k=0}^{\infty} \bar{\omega}^{k+1} w^k \\ &= -\sum_{\mu=-\infty}^{\infty} \sum_{n=-\infty}^{\mu} a_n \bar{\omega}^{\mu+1-n} w^\mu. \end{aligned} \tag{6}$$

The coefficient of w^{-1} is $\sum_{n=0}^{\infty} a_n\omega^n$ in (5) and $-\sum_{n=-\infty}^{-1} a_n\bar{\omega}^{-n}$ in (6). Thus, by the residue theorem, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma p'(z,0)p'(z,\zeta)\epsilon q(\phi(z))dz \\ &= \epsilon \sum_{n=0}^{\infty} a_n\omega^n + \epsilon \sum_{n=-\infty}^{-1} a_n\bar{\omega}^{-n} \\ &= \epsilon \sum_{n=-\infty}^{\infty} a_n\omega^n - \epsilon \sum_{n=-\infty}^{-1} a_n\omega^n + \epsilon \sum_{n=-\infty}^{-1} a_n(1/\bar{\omega})^n \\ &= \frac{\epsilon q(\omega)}{\omega f'(\omega)} - \epsilon S(\omega) + \epsilon S(1/\bar{\omega}). \end{aligned}$$

And with this, equation (4) becomes

$$\delta g(\zeta, 0) = \operatorname{Re}\left\{\frac{\epsilon q(\omega)}{\omega f'(\omega)} - \epsilon S(\omega) + \epsilon S(1/\bar{\omega})\right\} + O(\epsilon^2),$$

where we conjugate $S(1/\bar{\omega})$ so that the expression in braces is analytic.

Let $\phi_1^*(z)$ map D^* schlichtly onto $|w|<1$ such that $\phi_1^*(0)=0$. Now $g(\zeta, 0)=-\log|\phi(z)|$ so

$$\begin{aligned}\log|\phi_1^*(\zeta)| - \log|\phi(\zeta)| &= \delta g(\zeta, 0)\\ &= \mathrm{Re}\{K\} + O(\epsilon^2),\end{aligned}$$

where $K=\epsilon q(\omega)/\omega f'(\omega)-\epsilon S(\omega)+\epsilon\bar{S}(1/\bar{\omega})$. Completing this analytically and taking exponentials we have

$$\phi_1^*(\zeta)e^{-ic} - \phi(\zeta) = -\phi(\zeta)K + O(\epsilon^2),$$

where c is a real constant. Let $\phi^*(\zeta)=\phi_1^*(\zeta)e^{-ic}$ and let f^* be the inverse of ϕ^*. Then

$$\phi^*(\zeta) - \phi(\zeta) = -\phi(\zeta)K + O(\epsilon^2)$$

and since $f^*(\omega^*)=f^*(\phi^*(\zeta))=\zeta=f(\omega)$,

$$\begin{aligned}f(\omega) = f^*(\omega^*) &= f^*(\omega - \omega K + O(\epsilon^2))\\ &= f^*(\omega) - f^{*\prime}(\omega)\omega K + O(\epsilon^2)\\ &= f^*(\omega) - f'(\omega)\omega K + O(\epsilon^2),\end{aligned}$$

since $f^{*\prime}(\omega)=f'(\omega)+O(\epsilon^2)$. This completes the proof since ω was arbitrary in $|w|<1$, i.e.,

$$f^*(w) = f(w) + \epsilon q(w) - \epsilon w f'(w)S(w) + \epsilon w f'(w)\bar{S}(1/\bar{w}) + O(\epsilon^2)$$

is valid in $|w|<1$.

References

1. G. M. Golusin, *Geometrische Funktionentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

2. M. M. Schiffer, *Applications of variational methods in the theory of conformal mapping*, Proceedings of Symposia in Applied Mathematics, McGraw-Hill, New York, 1958.

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