ON COMMUTATORS OF OPERATORS ON HILBERT SPACE

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1. In this note we first generalize a result of P. R. Halmos [3] concerning commutators of (bounded) operators on Hilbert space. Then we obtain some partial results on a problem of commutators in von Neumann algebras which is closely related to another problem raised by Halmos in [4]. Let \( \mathcal{H} \) be any (infinite-dimensional) Hilbert space, and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded operators on \( \mathcal{H} \). We follow Halmos [3] in calling a subspace \( \mathcal{K} \subset \mathcal{H} \) large if \( \mathcal{K} \) contains infinitely many orthogonal copies of \( \mathcal{K} \oplus \mathcal{K} \). Halmos proved in [3] that any operator in \( \mathcal{L}(\mathcal{H}) \) with a large reducing null space is a commutator (of two bounded operators in \( \mathcal{L}(\mathcal{H}) \)). We generalize this to

**Theorem 1.** Any operator in \( \mathcal{L}(\mathcal{H}) \) which has a large null space is a commutator.

The construction involved in the proof of this theorem is a generalization of Halmos' construction in [3], and our construction actually yields a slightly more general result than Theorem 1. This more general result does not admit a nice formulation on nonseparable spaces, but on separable spaces it is easy to describe.

**Theorem 2.** Let \( \mathcal{H} \) be a separable Hilbert space and let \( \mathcal{K} = \mathcal{K} \oplus \mathcal{K} \).
If this decomposition of \( \mathcal{K} \) is used to write every operator \( T \in \mathcal{L}(\mathcal{H}) \) as a 2×2 operator matrix

\[
T = \begin{pmatrix} A & C \\ B & D \end{pmatrix},
\]

where the entries are operators on \( \mathcal{K} \), then every operator \( T \in \mathcal{L}(\mathcal{H}) \) of the form

(1) \[
T = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix},
\]

where \( C \) is a compact operator, is a commutator.

For separable spaces, Theorem 1 is a special case (\( C = 0 \)) of Theorem 2, and the proof of Theorem 1 for nonseparable spaces is an easy modification of the proof of that theorem for separable spaces. Thus we confine ourselves to proving Theorem 2.
Proof of Theorem 2. Let \( \{ E_i \}_{i=1}^\infty \) be any countable collection of mutually orthogonal projections \( E_i \in \mathcal{L}(\mathcal{K}) \) such that the sum of the \( E_i \) is the identity operator on \( \mathcal{K} \) and such that the range of each \( E_i \) is an infinite-dimensional subspace of \( \mathcal{K} \). Each \( E_i \) gives rise to a projection \( F_i \in \mathcal{L}(\mathcal{K}) \) defined by

\[
F_i = \begin{pmatrix} 0 & 0 \\ 0 & E_i \end{pmatrix}, \quad i = 1, 2, \cdots ,
\]

and if \( F_0 \in \mathcal{L}(\mathcal{K}) \) is defined as

\[
F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

then the \( F_i, i = 0, 1, 2, \cdots \), are mutually orthogonal projections on \( \mathcal{K} \) whose sum is the identity operator on \( \mathcal{K} \). Furthermore, the \( F_i \) are mutually equivalent in the sense of Murray-von Neumann in the von Neumann (v.N.) algebra \( \mathcal{L}(\mathcal{K}) \). Thus the \( F_i \) together with the implementing partial isometries form a complete set of matrix units for \( \mathcal{L}(\mathcal{K}) \), and we can use this set of matrix units to regard \( \mathcal{L}(\mathcal{K}) \) as an infinite matrix algebra. More precisely, it follows from [2, Proposition 5, p. 27], that \( \mathcal{L}(\mathcal{K}) \) is unitarily equivalent to the v.N. algebra \( \mathcal{A} \) of all \( \mathbb{K}_0 \times \mathbb{K}_0 \) matrices with entries from the v.N. algebra \( F_0 \mathcal{L}(\mathcal{K}) F_0 \cong \mathcal{L}(\mathcal{K}) \) which act as operators on the Hilbert space \( \mathcal{K}_1 = \mathcal{K} \otimes \mathcal{K} \otimes \cdots \) (Recall that \( F_0(\mathcal{K}) = \mathcal{K} \otimes 0 \)). Thus we can and do work with the infinite matrices of \( \mathcal{A} \) instead of the \( 2 \times 2 \) matrices of \( \mathcal{L}(\mathcal{K}) \). It is easy to see that any operator in \( \mathcal{L}(\mathcal{K}) \) of the form

\[
\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}
\]

is carried by the isomorphism between \( \mathcal{L}(\mathcal{K}) \) and \( \mathcal{A} \) onto an operator in \( \mathcal{A} \) of the form

\[
X = \begin{pmatrix} A & C_1 & C_2 & C_3 & \cdots \\ B_1 & 0 & 0 & 0 & \cdots \\ B_2 & 0 & 0 & \cdots & \cdots \\ B_3 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.
\]

A matrix calculation shows that formally \( X \) is the commutator \( X = SR - RS \) where
\[
S = \begin{pmatrix}
-B_1 & A & C_1 & C_2 & C_3 & \cdots \\
-B_2 & 0 & A & C_1 & C_2 & \cdots \\
-B_3 & 0 & 0 & A & C_1 & \cdots \\
& 0 & 0 & 0 & . & . \\
& 0 & 0 & 0 & 0 & . \\
& & & & & \\
& & & & & \\
\end{pmatrix},
\]

and

\[
R = \begin{pmatrix}
0 & 0 & 0 & . & . \\
1 & 0 & 0 & . & . \\
0 & 1 & 0 & . & . \\
& 0 & 1 & . & . \\
& & 0 & . & . \\
& & & . & . \\
\end{pmatrix}
\]

It is obvious that \( R \) represents a bounded operator in \( A \), and thus to complete the proof it suffices to show that if \( C \) is compact then the collection \( \{E_i\}_{i=1}^\infty \) can be chosen in such a way as to ensure that \( S \) represents a bounded operator. Since \( X \) is a bounded operator and since, obviously, the matrix

\[
\begin{pmatrix}
0 & A & 0 & . \\
0 & A & 0 & . \\
0 & A & 0 & . \\
& & & . \\
& & & . \\
\end{pmatrix}
\]

represents a bounded operator, it is easy to see that it suffices to show that the collection \( \{E_i\} \) can be chosen so that the "Toeplitz" matrix

\[
Y = \begin{pmatrix}
C_1 & C_2 & C_3 & . \\
0 & C_1 & C_2 & . \\
& 0 & C_1 & . \\
& & 0 & . \\
& & & . \\
\end{pmatrix}
\]

represents a bounded operator. That this can be done follows from the following sequence of lemmas.
Lemma 1.1. If $C$ is a compact operator on the separable Hilbert space $\mathcal{H}$, and $C = UP$ is the polar decomposition of $C$, then there exists a sequence of mutually orthogonal infinite-dimensional subspaces $\mathcal{H}_1, \mathcal{H}_2, \ldots \subset \mathcal{H}$ such that

(a) $\sum_{i=1}^{\infty} \mathcal{H}_i = \mathcal{H}$;

(b) each $\mathcal{H}_i$ is a reducing subspace for $P$, so that the linear manifolds $\{C(\mathcal{H}_i)\}$ are mutually orthogonal;

(c) for each $i \geq 2$, $\|P \mathcal{H}_i\| = \|C \mathcal{H}_i\| < 1/2^i$.

Proof. If $P$ has an infinite-dimensional null space $\mathcal{N}$, take $\mathcal{H}_1$ to be the direct sum of $\mathcal{N} \oplus \mathcal{N}$ and a sufficiently large subspace $\mathcal{M} \subset \mathcal{H}$ to ensure that $\mathcal{H}_1$ is infinite-dimensional. Arrange it so that $\mathcal{N} \oplus \mathcal{M}$ remains infinite-dimensional, and write $\mathcal{N} \oplus \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$, where $\mathcal{M}_2, \mathcal{M}_3, \ldots$ are also infinite-dimensional. If $\mathcal{N}$ is finite-dimensional, then

$$P = \sum_{i=1}^{\infty} \alpha_i E_i,$$

where each $\alpha_i$ is a positive scalar, $\{\alpha_i\} \to 0$, and the $E_i$ are mutually orthogonal projections on finite-dimensional spaces. Now the problem is essentially that of partitioning a countable set into a countable union of disjoint infinite subsets, maintaining some care so that (c) will be satisfied. We omit further details of that argument.

Lemma 1.2. With $C, \mathcal{H}$ and the sequence $\{\mathcal{H}_i\}$ as in Lemma 1.1, for each positive integer $i$, let $E_i \in \mathcal{L}(\mathcal{H})$ be the projection on $\mathcal{H}_i$. If the collection $\{E_i\}_{i=1}^{\infty}$ is used to determine a unitary equivalence between $\mathcal{L}(\mathcal{H})$ and $\mathcal{A}$ as above, then the operators $C_i \in \mathcal{L}(\mathcal{H})$ which appear in the matrix $X$ have mutually orthogonal ranges and in addition satisfy

$$\sum \|C_i\|^2 < \infty.$$

Proof. By definition [2, Proposition 5, p. 27], $C_i$ is the restriction to $\mathcal{N} \oplus 0 \subset \mathcal{H}$ of an operator $F_i T U_i \in \mathcal{L}(\mathcal{H})$, where $T$ is as in (1) and $U_i \in \mathcal{L}(\mathcal{H})$ is a partial isometry with initial space $\mathcal{N} \oplus 0 \subset \mathcal{H}$ and final space $0 \oplus \mathcal{N} \subset \mathcal{H}$. Each $U_i$ obviously has a $2 \times 2$ matrix

$$U_i = \begin{pmatrix} 0 & 0 \\ V_i & 0 \end{pmatrix},$$

where $V_i \in \mathcal{L}(\mathcal{H})$ satisfies $V_i V_i^* = E_i$ and $V_i^* V_i = 1_{\mathcal{H}}$.

By multiplying the appropriate $2 \times 2$ matrices we obtain

$$F_i T U_i = \begin{pmatrix} CV_i & 0 \\ 0 & 0 \end{pmatrix}. $$

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Thus for \( x, y \in \mathcal{K} \oplus 0 \) and \( i \neq j \), \( (C_i \alpha, C_j \gamma) = (F_\alpha T U \alpha, F_\gamma T U \gamma) \) \( = (CV \alpha, CV \gamma) = 0 \) since \( V \alpha \in \mathcal{K}_i \) and \( V \gamma \in \mathcal{K}_j \). This proves that \( C_i \) and \( C_j \) have orthogonal ranges. Also, for \( i \geq 2 \) and \( \|x\| = 1 \), \( \|C_i x\|^2 = (CV \alpha, CV \alpha) = \|CV \alpha\|^2 \leq \|C_i \alpha\|^2 \leq 1/2^i \), so that clearly
\[
\sum_i \|C_i\|^2 < \infty.
\]

To complete the proof of Theorem 2, it now suffices to show that if the operators \( C_i \in \mathcal{L}(\mathcal{K}) \) which appear in the matrix \( Y \) have mutually orthogonal ranges and satisfy
\[
\sum_i \|C_i\|^2 < \infty,
\]
then \( Y \) is bounded. An easy computation which we omit shows that indeed this is the case, and in fact
\[
\|Y\| \leq \sum_i \|C_i\|^2.
\]

As an immediate corollary of Theorem 1 we obtain

**Corollary 1.3 (Halmos).** Any operator \( A \) on a Hilbert space \( \mathcal{K} \) is the sum of two commutators.

**Proof.** Write \( \mathcal{K} = \mathcal{K} \oplus \mathcal{K} \) and write \( A \) as the sum of two operators each of which vanishes on one copy of \( \mathcal{K} \).

**Conjecture.** The author conjectures that Theorem 2 remains true if the restriction of compactness is removed from the operator \( C \).

2. In [4], Halmos raised the question of whether every operator on a separable Hilbert space which is not a scalar modulo the compact operators is a commutator. If the answer to this question is yes, and we denote the closed ideal of compact operators by \( \mathcal{C} \), then the \( \mathcal{C}^* \)-algebra \( \mathcal{L}(\mathcal{K})/\mathcal{C} \) has the following property:

(S) Every nonscalar element of the algebra is a commutator of two elements in the algebra.

Calkin [1] imbeds \( \mathcal{L}(\mathcal{K})/\mathcal{C} \) in a v.N. algebra \( \mathcal{C} \) (acting on a non-separable Hilbert space), and thus a related question is whether \( \mathcal{C} \), or for that matter any v.N. algebra, has property (S). A partial answer is given by the following theorem.

**Theorem 3.** Every v.N. algebra \( \mathcal{A} \) which is not a factor of type III contains a projection \( E \neq 1 \) which is not a commutator in \( \mathcal{A} \); thus, no such \( \mathcal{A} \) has property (S).

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1 See Remark (3) at the end of the paper.
Proof. If $A$ is not a factor, it follows from Kleinecke's result in [6] that every nonzero central projection fails to be a commutator in $A$, so that it suffices to consider the case that $A$ is a factor. If $A$ is a finite factor, then $A$ possesses a numerical-valued trace, and thus any projection with nonzero trace fails to be a commutator in $A$. If $A$ is a factor of type $I_m$, then $A$ is algebraically isomorphic to the algebra of all bounded operators on some Hilbert space $\mathcal{H}$ and thus $A$ contains the proper closed ideal $\mathcal{S}$ of compact operators on $\mathcal{H}$. It follows that any projection of the form $1-E$ where $E$ is a finite-dimensional projection cannot be a commutator in $A$. (If $1-E$ were a commutator in $A$, then the identity element $1+\mathcal{S}$ of the Banach algebra $A/\mathcal{S}$ would be a commutator in $A/\mathcal{S}$, which is impossible [4].) Finally, if $A$ is a factor of type $II_\infty$, let $\mathcal{F}$ be the subset of $A$ consisting of all elements which are of "finite rank" in the sense of [7, Definition 1.2.1, p. 97]. It follows from [7, Lemma 1.2.1, p. 97] that $\mathcal{F}$ is a two-sided ideal in $A$ and thus the uniform closure $\mathcal{F}_0$ of $\mathcal{F}$ is a proper closed two-sided ideal in $A$. If $E$ is any finite projection in $A$, then $E \in \mathcal{F}$ and again $1-E$ cannot be a commutator in $A$, which completes the argument.

Whether there is a type III factor with property (S) or not, the author does not know. However, the following theorem indicates that perhaps every type III factor has property (S).

Theorem 4. If $A$ is a factor of type III, then every $A \in A$ which has a nontrivial null space is a commutator in $A$. In particular, every projection $P \neq 1$ in $A$ is a commutator in $A$.

Proof. One knows that the projection $E$ on the null space of $A$ is an element of $A$. It follows from repeated application of Lemma 4.12 of [5] and the fact that all nonzero projections in $A$ are equivalent that there exists a countable family $\{F_i\}$ of mutually orthogonal, equivalent projections in $A$ such that

$$\sum_i F_i = E.$$ 

If we adjoin $1-E$ to the family $\{F_i\}$ we obtain a countable family of mutually orthogonal, equivalent projections in $A$ whose sum is 1. An application of [2, Proposition 5, p. 27] yields a unitary isomorphism of $A$ onto the v.N. algebra $B$ of all $\mathbb{K}_0 \times \mathbb{K}_0$ operator matrices with entries from the algebra $(1-E)A(1-E)$, and under this isomorphism...
$A$ is carried onto a matrix of the form of $X$ in Theorem 2, where $C_1 = C_2 = \cdots = 0$. But then $X = SR - RS$, just as in Theorem 2.

3. **Remarks.** (1) I wish to express my appreciation to Professor Paul Halmos for stimulating my interest in commutators and to Don Deckard for many interesting conversations on the subject.

   (2) Calkin conjectured in [1] that the v.N. algebra which he constructed to contain $\mathcal{L}(\mathcal{C})/\mathcal{C}$ is a factor of type III. Theorems 3 and 4 lend support to that conjecture.

   (3) **Added in proof.** Since this paper was written, Arlen Brown and the author have completely settled the question of which operators on Hilbert space are commutators. Theorem 1 of the present paper proved useful in that connection. (See: Structure theorem for commutators of operators, Arlen Brown and Carl Pearcy, Bull. Amer. Math. Soc. 70 (1964), 779–780.) We have also proved that when $\mathcal{C}$ is separable, every factor of type III on $\mathcal{C}$ has property (S).

**Bibliography**


