A CONVOLUTION EQUATION ON A COMPACT INTERVAL

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1. Introduction. We present two theorems which give sufficient conditions for the existence of a solution, s, of the convolution equation \( g(x) = \int_J W(x-y) \, ds(y) \) (briefly, \( g = W * s \)). Here \( J \) is, say, the compact real interval \([0, 1]\), \( g \) is an \( L^2(J, dx) \) function, and \( W \) is a complex-valued function on \([-1, 1]\) with the following property: The operator \( [D^2W *]: f \rightarrow D^2 W * f = \int_{-1}^1 f(x-y) \, dD^2W(y) \) maps \( L^2(J, dx) \) into itself, where differentiation, \( D \), and convolution, \( * \), are distribution operators as defined by L. Schwartz \([1]\). The guaranteed solution, \( s \), will be a distribution with support in \( J \), of order depending on the differentiability of \( W \).

In recent treatments of the convolution equation over a compact interval by Shinbrot and Akutowicz \([2]\), \([3]\), \([4]\), hypotheses are made on the Fourier transform of \( W \), \( \mathcal{F}W \), which place restrictions on the real zeros of \( \mathcal{F} W \) and exclude such kernels as \( \mathcal{F}^{-1} \sin^2 x/x^2 \). Now, the values of \( W \) outside a neighborhood of \([-1, 1]\) do not influence \( W * s \) on \( J \) if the distribution \( s \) has support in \( J \). The existence and nature of solutions of the convolution equation depends in principle, therefore, only upon \( W \) as it is defined in a neighborhood of \([-1, 1]\). Although in this paper we do not strictly generalize the results of \([2]\), \([3]\), \([4]\), we do, in effect, restrict the values of \( W \) only on a neighborhood of \([-1, 1]\).

We employ Chover's idea \([5]\), of assuming \( W \) not differentiable at 0. \( D^2W \) then has a point mass at 0, and the operator \( [D^2W *] \) can be written as the sum of two operators, one of which is the identity, \( I \). In our theorems we assume that \( D^2W = k - \delta_0 - \psi \) on \([-1, 1]\), where \( k \) is a function of bounded variation and \( \psi \) is a measure of variation less than 1. Results for kernels \( W \) such that \( D^nW \) has this form for some positive integer \( n \) follow immediately as corollaries.

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2. Definitions. We use the term positive definite in the following sense. We call a continuous complex-valued function \( W \) on the real
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line, \( R \), positive definite if for every nonzero Radon measure \( m \) of compact support on \( R \), \( \int W(x-y) \, dm(x) \, d\tilde{m}(y) > 0 \). We use the equivalent form, \( W \ast m \ast \tilde{m}(0) > 0 \), where \( \tilde{m} \) denotes the reflection of \( m \) about 0.

In expressions involving the distribution operators \( D \) and \( * \), we regard functions and measures as the associated distributions. We use several theorems found in [1] without specific reference. The primary one is that if \( f \) and \( g \) are distributions, at least one of which has compact support, then \( D(f \ast g) = Df \ast g = f \ast Dg \).

We regard functions in \( L^2(J, dx) \) as identically 0 outside of \( J \). \( \delta_x \) denotes the measure which is 0 except at \( x \) where it has mass 1. Hence \( [\delta_0 \ast] \) is the identity operator, \( I \). By \( [D^2W \ast] \) we mean the operator on \( L^2(J, dx) \) which maps a function \( f \) into the function which is \( D^2W \ast f \) on \( J \) and zero elsewhere. If \( H \subset J \), \( J-H = \{ x: x \in J \) and \( x \notin H \} \). If \( T \) is an operator, \( R(T) = \text{range of } T \), and \( N(T) = \text{null space of } T \).

In Theorem 2 there appear distributions which are functions on the interior of \( J \) and are extendable to functions of bounded variation on \( J \). We call the class of such distributions \( B.V.(J) \).

3. Existence theorems and proofs.

THEOREM 1. Let \( W \) be a positive definite function on \([-1, 1]\) such that \( D^2W = k-\delta_0-\psi \), where \( k \) is a function of bounded variation on \([-1, 1]\) and \( \psi \) is a measure on \([-1, 1]\) of variation less than 1. Then for any \( g \in L^2(J) \) there exists \( f \in L^2(J) \) such that \( g = W \ast D^2f \) on \( J \). If \( g \) is a continuous function of bounded variation on \( J \) then \( f \) is also.

In the proof we use the following lemma.

LEMMA 1. If \( g = k \ast f - f - \psi \ast f \) on \( J \), where \( f \in L^2(J) \) and \( g \) is a continuous function of bounded variation on \( J \), then \( f \) is a continuous function of bounded variation on \( J \).

PROOF. Since the convolution of two \( L^2 \) functions is continuous and \( k \) has bounded variation, \( k \ast f \) is a continuous function of bounded variation on \( J \) and so is \( k \ast f - g = f + \psi \ast f \). As is well known, \( \text{var } \psi < 1 \) implies that \( [\psi \ast] \) is an operator of norm \( < 1 \) on \( L^2(J) \) as well as on \( C_J \). There exists \( f_1 \in C_J \) such that \( f_1 + \psi \ast f_1 = f + \psi \ast f \) on \( J \). Since \( f_1 \) and \( f \) are both in \( L^2(J) \), \( f_1 = f \) and \( f \in C_J \).

To show that \( f \) is a function of bounded variation on \( J \) it suffices to show that \( \sum_{i=1}^n \left| f(x_i) - f(x_{i-1}) \right| \) is uniformly bounded for any \( n \) and for any choice of \( \{ x_i, i=0, \cdots, n \} \subset J \). Let \( v = \text{var}_J(f + \psi \ast f) \). Fix \( n \) and choose \( \{ x_i, i=0, \cdots, n \} \subset J \) such that
is maximized. Then

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq v + \sum_{i=1}^{n} \int_{-1}^{1} |f(x_i - y) - f(x_{i-1} - y)| \, d\psi(y)$$

$$\leq v + \text{var} \cdot \sup_{y} \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)|.$$

Since $g = 0$ on the complement of $J$,

$$\sup_{y} \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)| \leq \max_{\{x_i\} \subset J} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$+ \sup |f|.$$

Therefore

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq (v + \text{var} \cdot \sup |f|) / (1 - \text{var} \psi).$$

This completes the proof of the lemma.

To prove Theorem 1 we first note that since $W$ is positive definite, it is Hermitian, and $[D^2W \ast] = [(k - \delta_0 - \psi) \ast]$ is a self-adjoint operator on $L^2(J)$. The closure of $R[D^2W \ast]$ is therefore the orthogonal complement of $N[D^2W \ast]$. We prove the first statement of the theorem by showing that $R[D^2W \ast]$ is closed and that $N[D^2W \ast] = 0$. Then $R[D^2W \ast] = L^2(J)$ and for any $g \in L^2(J)$ there exists $f \in L^2(J)$ such that $g = D^2W \ast f = W \ast D^2f$.

If $T$ and $S$ are continuous linear operators on $L^2$ such that $T$ is completely continuous and $S^{-1}$ is continuous then $R[T - S]$ is closed. This can be shown by an argument closely analogous to that for the familiar case where $S$ is the identity. Since $k(x - y)$ is an element of $L^2([-1, 1] \times [-1, 1])$, $[k \ast]$ is a completely continuous operator on $L^2(J)$. $[(\delta_0 + \psi) \ast] = I + [\psi \ast]$ has a continuous inverse. Therefore $R[[k \ast] - [(\delta_0 + \psi) \ast]]$ is closed.

Suppose $f \in N[D^2W \ast]$, i.e., $k \ast f - f - \psi \ast f = 0$ on $J$. By Lemma 1, $Df$ is a measure on $R$. Since $W$ is positive definite, $W \ast Df \ast (Df)^{-}(0) > 0$ unless $Df \equiv 0$. $W \ast Df \ast (Df)^{-}(0) = -(D^2W \ast f) \ast f(0)$ $= -\int_{-\gamma}(D^2W \ast f)(-\gamma)\bar{f}(-\gamma)dy$ since the support of $\bar{f}$ is contained in $-J$. But since $D^2W \ast f = 0$ on $J$, $W \ast Df \ast (Df)^{-}(0) = 0$, and $Df \equiv 0$. $f$ has support in $J$. Therefore $Df \equiv 0$ implies $f \equiv 0$.

Suppose $g$ is a continuous function of bounded variation on $J$, and let $f$ be the element of $L^2(J)$ such that $g = D^2W \ast f = k \ast f - f - \psi \ast f$. 

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By Lemma 1, $f$ is a continuous function of bounded variation on $J$.

From this theorem we get, immediately:

**Corollary 1.** If $D^nW$ satisfies the hypothesis of Theorem 1, i.e., $D^nW$ is a positive definite function on $[-1, 1]$ such that $D^{n+2}W = k - \delta_0$ where $\delta_0$ as stated, then for any $g \in L^2(J)$ there exists $f \in L^2(J)$ such that $g = W * D^{n+2}f$. If $g$ is a continuous function of bounded variation on $J$, then $D^{n+2}f$ is an $(n+1)$st order distribution with support in $J$.

**Theorem 2.** Let $H$ be a closed interval contained in the interior of $J$. Let $T$ be the self-adjoint operator on $L^2(J)$ defined by $Tf = (k - \delta_0) * f$, where $k$ is a complex-valued, Hermitian function of bounded variation on $[-1, 1]$. Then

(a) For any $g \in L^2(H)$ there exists $f \in L^2(J)$ such that $g = Tf$ on $H$.

(b) If $g$ is a function of bounded variation on $H$, there exists $f \in L^2(J)$ of bounded variation such that $g = Tf$ on $H$.

(c) If $Dg$ is in $B.V.(H)$, there exists $f \in L^2(J)$ such that $Df$ is in $B.V.(J)$ and $g = Tf$ on $H$.

In the proof of Theorem 2 we use the following lemma.

**Lemma 2.** If $g = k * f - f$ on $J$, where $Dg$ is in $B.V.(J)$ and $f$ is a measure with support in $J$ then $Df$ is in $B.V.(J)$.

**Proof.** Since $k * f$ and $g$ are both functions of bounded variation on $J$, $f$ is a function of bounded variation on $J$ and therefore on $R$. Hence, $Df$ is a measure, and $D(k * f) = k * Df$ is in $B.V.(J)$. Therefore so is $Df$, which proves the lemma.

To prove Theorem 2 we first note that $T$ is a self-adjoint operator on $L^2(J)$ and that $R(T)$ is closed and is the orthogonal complement of $N(T)$. Given $g \in L^2(H)$, we will construct an element of the orthogonal complement of $N(T)$ which equals $g$ on $H$.

Since $[k *]$ is a completely continuous operator on $L^2(J)$, $N(T)$ is finite-dimensional. Choose $\{f_i, i = 1, \ldots, n\}$, linearly independent on $J$ and spanning $N(T)$. Then the $f_i$ are linearly independent on $J - H$, as we now show.

Consider any $f = \sum_{i=1}^{n} c_i f_i, f \neq 0$ on $J$. Suppose $f = 0$ on $J - H$. Then $f = 0$ on $H = R - H$. $f$ is an element of $N(T)$, i.e., $k * f - f = 0$ on $J$. Lemma 2, where $g = 0$, implies that $Df$ is in $B.V.(J)$. Therefore $D^2f$ is a measure on the interior of $J$. $D^3f = 0$ on $H$. Therefore $D^n f$ is a measure on $R$, with support in $J$. By Lemma 2, again, $D^n f$ is in $B.V.(J)$. $D^n f = 0$ on $H$. Therefore $D^n f$ is a measure on $R$. Similarly, $D^2 f$ is a measure on $R$ implies $D^{2n+2} f$ is a measure on $R$, and so by induction $D^n f$ is a measure for all $n$. For any distribution $h$, $Dh$ is a function implies $h$ is a continuous function, and $Dh$ is a continuous...
function implies $h$ is differentiable and $Dh = h^{(1)}$. Therefore $D^{*}f = f^{(n)}$, $f \in C^{n}$, and $f^{(n)} \in L^{2}(J)$ for all $n$. $f^{(n)}$ is, consequently, an element of the domain of $T$, and $Tf^{(n)} = TD^{*}f = (k - \delta_{0}) \star D^{*}f = D^{ n}[(k - \delta_{0}) \star f]$, which is 0 on the interior of $J$ because $(k - \delta_{0}) \star f = 0$ on $J$. Since $f$ is in $C^{n}$ on all of $R$, so is $D^{ n}[(k - \delta_{0}) \star f]$. Therefore $Tf^{(n)} = 0$ on $J$, i.e., $f^{(n)} \in N(T)$ for all $n$. But since $N(T)$ is finite-dimensional, there exists an integer $m$ and constants $b_{j}, j = 0, \ldots, m$, not all zero, such that $\sum_{j=0}^{m} b_{j} f^{(n)} = 0$ on $J$. Consequently $f$ is analytic on the interior of $J$ and cannot be 0 on $J - H$. It follows that the $f_{i}$ are linearly independent on $J - H$ as asserted.

The $f_{i}$ can be chosen so that their restrictions to $J - H$ form an orthonormal set. For instance, if $g_{i} = \sum a_{i,j} g_{j}, i = 1, \ldots, n$, is the Gram-Schmidt orthogonalization of the $f_{i}$ restricted to $J - H$, then the $g_{i}$, considered as functions on $J$ still span $N(T)$, since each $f_{i}$ is a linear combination of the $g_{i}$. We assume that the $f_{i}$ are so chosen.

Define
\[ G(x) = g(x) \quad \text{for } x \in H, \quad G(x) = \sum_{i=1}^{n} b_{i}f_{i}(x) \quad \text{for } x \in J - H, \]

where $b_{i} = -\int_{H} f_{i}(x) g(x) \, dx$. Then $G \in L^{2}(J)$.

\[
\int_{J} f_{i}(x) G(x) \, dx = \int_{J-H} f_{i}(x) \left( \sum_{j=1}^{n} b_{j}f_{j}(x) \right) \, dx + \int_{H} f_{i}(x) g(x) \, dx
= -b_{i} + b_{i} = 0,
\]

for each $i = 1, \ldots, n$. Therefore $G$ is in the orthogonal complement of $N(T)$ which as noted above is $R(T)$, i.e., there exists $f \in L^{2}(J)$ such that $G = Tf$ on $J$. Since $g = G = Tf$ on $H$, (a) is proved.

(b) Suppose $g$ is a function of bounded variation on $H$. If $f_{i} \in N(T)$, i.e., $k \star f_{i} = f_{i}$ on $J$, then $f_{i}$ is a function of bounded variation. Therefore $G$, as defined above, is a function of bounded variation, and $Tf = k \star f - f = G$ on $J$ implies $f$ is a function of bounded variation on $J$ and therefore on $R$.

(c) Suppose $Dg$ is in $B.V.(H)$. If $f_{i} \in N(T)$ then Lemma 2 implies that $Df_{i}$ is in $B.V.(J)$. Therefore $G$ as defined above is such that $DG$ can fail to be in $B.V.(J)$ only if $G$ has discontinuities at the endpoints, $c$ and $d$, of $H$. To remedy this we will alter the definition of $G$ on $J - H$. Let $h(x)$ be a function on $J - H$ such that

(i) $h$ is in the orthogonal complement of $N(T)$,

(ii) $Dh$ is a function of bounded variation on $J - H$,

(iii) $h(c) = g(c) - \sum_{i=1}^{n} b_{i}f_{i}(c)$, $h(d) = g(d) - \sum_{i=1}^{n} b_{i}f_{i}(d)$.

Define
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\[ G(x) = \sum_{i=1}^{n} b_i f_i(x) + k(x) \text{ for } x \in J - H, \quad G(x) = g(x) \text{ for } x \in H. \]

Then \( DG \) is in \( \text{B.V.}(J) \), and \( G \) is in the orthogonal complement of \( N(T) \). There exists \( f \in L^4(J) \) such that \( G = k * f - f \) on \( J \). Lemma 2 implies that \( Df \) is in \( \text{B.V.}(J) \).

We obtain immediately from Theorem 2.

**Corollary 2.** If \( W \) is a function on \([-1, 1]\) such that \( D^n W = k - \delta_0 \) for some positive integer \( n \), where \( k \) is as in Theorem 2, then for every \( g \in L^4(H) \) there exists \( f \in L^4(J) \) such that \( g = W * D^n f \) on \( H \).

**References**


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