A CONVOLUTION EQUATION ON A COMPACT INTERVAL

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1. Introduction. We present two theorems which give sufficient conditions for the existence of a solution, $s$, of the convolution equation $g(x) = \int_J W(x - y) \, ds(y)$ (briefly, $g = W * s$). Here $J$ is, say, the compact real interval $[0, 1]$, $g$ is an $L^2(J, dx)$ function, and $W$ is a complex-valued function on $[-1, 1]$ with the following property: The operator $[D^2 W *]: f \rightarrow D^2 W * f = \int_{-1}^1 f(x - y) \, dD^2 W(y)$ maps $L^2(J, dx)$ into itself, where differentiation, $D$, and convolution, $*$, are distribution operators as defined by L. Schwartz [1]. The guaranteed solution, $s$, will be a distribution with support in $J$, of order depending on the differentiability of $W$.

In recent treatments of the convolution equation over a compact interval by Shinbrot and Akutowicz [2], [3], [4], hypotheses are made on the Fourier transform of $W$, $\mathcal{F} W$, which place restrictions on the real zeros of $\mathcal{F} W$ and exclude such kernels as $\mathcal{F}^{-1} \sin^2 x/x^2$. Now, the values of $W$ outside a neighborhood of $[-1, 1]$ do not influence $W * s$ on $J$ if the distribution $s$ has support in $J$. The existence and nature of solutions of the convolution equation depends in principle, therefore, only upon $W$ as it is defined in a neighborhood of $[-1, 1]$. Although in this paper we do not strictly generalize the results of [2], [3], [4], we do, in effect, restrict the values of $W$ only on a neighborhood of $[-1, 1]$.

We employ Chover's idea [5], of assuming $W$ not differentiable at $0$. $D^2 W$ then has a point mass at $0$, and the operator $[D^2 W *]$ can be written as the sum of two operators, one of which is the identity, $I$. In our theorems we assume that $D^2 W = k - \delta - \psi$ on $[-1, 1]$, where $k$ is a function of bounded variation and $\psi$ is a measure of variation less than $1$. Results for kernels $W$ such that $D^2 W$ has this form for some positive integer $n$ follow immediately as corollaries.

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2. Definitions. We use the term positive definite in the following sense. We call a continuous complex-valued function $W$ on the real
line, \( R \), positive definite if for every nonzero Radon measure \( m \) of compact support on \( R \), \( \int W(x-y) \, dm(x) \, d\bar{m}(y) > 0 \). We use the equivalent form, \( W \ast m \ast \bar{m}(0) > 0 \), where \( \bar{m} \) denotes the reflection of \( m \) about 0.

In expressions involving the distribution operators \( D \) and \( \ast \), we regard functions and measures as the associated distributions. We use several theorems found in \([1]\) without specific reference. The primary one is that if \( f \) and \( g \) are distributions, at least one of which has compact support, then \( D(f \ast g) = Df \ast g = f \ast Dg \).

We regard functions in \( L^2(J, dx) \) as identically 0 outside of \( J \). \( \delta_x \) denotes the measure which is 0 except at \( x \) where it has mass 1. Hence \( [\delta_x \ast] \) is the identity operator, \( I \). By \([D^2W \ast] \) we mean the operator on \( L^2(J, dx) \) which maps a function \( f \) into the function which is \( D^2W \ast f \) on \( J \) and zero elsewhere. If \( H \subset J \), \( J - H = \{x: x \in J \text{ and } x \notin H\} \). If \( T \) is an operator, \( R(T) = \text{range of } T \), and \( N(T) = \text{null space of } T \).

In Theorem 2 there appear distributions which are functions on the interior of \( J \) and are extendable to functions of bounded variation on \( J \). We call the class of such distributions \( \text{B.V.}(J) \).

3. Existence theorems and proofs.

**Theorem 1.** Let \( W \) be a positive definite function on \([-1, 1]\) such that \( D^2W = k - \delta_0 - \psi \), where \( k \) is a function of bounded variation on \([-1, 1]\) and \( \psi \) is a measure on \([-1, 1]\) of variation less than 1. Then for any \( g \in L^2(J) \) there exists \( f \in L^2(J) \) such that \( g = W \ast D^2f \) on \( J \). If \( g \) is a continuous function of bounded variation on \( J \) then \( f \) is also.

In the proof we use the following lemma.

**Lemma 1.** If \( g = k \ast f - f - \psi \ast f \) on \( J \), where \( f \in L^2(J) \) and \( g \) is a continuous function of bounded variation on \( J \), then \( f \) is a continuous function of bounded variation on \( J \).

**Proof.** Since the convolution of two \( L^2 \) functions is continuous and \( k \) has bounded variation, \( k \ast f \) is a continuous function of bounded variation on \( J \) and so is \( k \ast f - g = f + \psi \ast f \). As is well known, \( \text{var } \psi < 1 \) implies that \([\psi \ast] \) is an operator of norm \(< 1 \) on \( L^2(J) \) as well as on \( C_J \). There exists \( f_1 \in C_J \) such that \( f_1 + \psi \ast f_1 = f + \psi \ast f \) on \( J \). Since \( f_1 \) and \( f \) are both in \( L^2(J) \), \( f_1 = f \) and \( f \in C_J \).

To show that \( f \) is a function of bounded variation on \( J \) it suffices to show that \( \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \) is uniformly bounded for any \( n \) and for any choice of \( \{x_i, i=0, \ldots, n\} \subset J \). Let \( v = \text{var}_J(f + \psi \ast f) \).

Fix \( n \) and choose \( \{x_i, i=0, \ldots, n\} \subset J \) such that
is maximized. Then

\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq v + \sum_{i=1}^{n} \int_{-1}^{1} |f(x_i - y) - f(x_{i-1} - y)| \, d\psi(y)
\]

\[
\leq v + \var \sup_y \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)|.
\]

Since \(g = 0\) on the complement of \(J\),

\[
\sup_y \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)| \leq \max_{\{x_i\} \subseteq J} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sup |f|.
\]

Therefore

\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq (v + \var \sup |f|)/(1 - \var \psi).
\]

This completes the proof of the lemma.

To prove Theorem 1 we first note that since \(W\) is positive definite, it is Hermitian, and \([D^2W \ast] = [(k - \delta_0 - \psi) \ast]\) is a self-adjoint operator on \(L^2(J)\). The closure of \(R[D^2W \ast]\) is therefore the orthogonal complement of \(N[D^2W \ast]\). We prove the first statement of the theorem by showing that \(R[D^2W \ast]\) is closed and that \(N[D^2W \ast] = 0\). Then \(R[D^2W \ast] = L^2(J)\) and for any \(g \in L^2(J)\) there exists \(f \in L^2(J)\) such that \(g = D^2W \ast f = W \ast D^2f\).

If \(T\) and \(S\) are continuous linear operators on \(L^2\) such that \(T\) is completely continuous and \(S\) is continuous then \(R[T - S]\) is closed. This can be shown by an argument closely analogous to that for the familiar case where \(S\) is the identity. Since \(k(x - y)\) is an element of \(L^2([-1, 1] \times [-1, 1])\), \([k \ast]\) is a completely continuous operator on \(L^2(J)\). \([(\delta_0 + \psi) \ast] = I + [\psi \ast]\) has a continuous inverse. Therefore \(R[[k \ast] - [(\delta_0 + \psi) \ast]]\) is closed.

Suppose \(f \in N[D^2W \ast]\), i.e., \(k \ast f - f - \psi \ast f = 0\) on \(J\). By Lemma 1, \(Df\) is a measure on \(R\). Since \(W\) is positive definite, \(W \ast Df \ast (Df)^{-0}(0) > 0\) unless \(Df = 0\). \(W \ast Df \ast (Df)^{-0}(0) = - (D^2W \ast f) \ast f(0) = - \int_{-J} (D^2W \ast f)(-y)f(-y) \, dy\) since the support of \(f\) is contained in \(-J\). But since \(D^2W \ast f = 0\) on \(J\), \(W \ast Df \ast (Df)^{-0}(0) = 0\), and \(Df = 0\).

Suppose \(g\) is a continuous function of bounded variation on \(J\), and let \(f\) be the element of \(L^2(J)\) such that \(g = D^2W \ast f = k \ast f - f - \psi \ast f\).
By Lemma 1, \( f \) is a continuous function of bounded variation on \( J \).
From this theorem we get, immediately:

**Corollary 1.** If \( D^nW \) satisfies the hypothesis of Theorem 1, i.e.,
\( D^nW \) is a positive definite function on \([-1, 1]\) such that \( D^{n+2}W = k - \delta_0 \)
\(-\psi \) as stated, then for any \( g \in L^2(J) \) there exists \( f \in L^2(J) \) such that
\( g = W * D^{n+2}f \). If \( g \) is a continuous function of bounded variation on \( J \),
then \( D^{n+2}f \) is an \((n+1)\)st order distribution with support in \( J \).

**Theorem 2.** Let \( H \) be a closed interval contained in the interior of \( J \).
Let \( T \) be the self-adjoint operator on \( L^2(J) \) defined by \( Tf = (k - \delta_0) * f \),
where \( k \) is a complex-valued, Hermitian function of bounded variation
on \([-1, 1]\). Then

(a) For any \( g \in L^2(H) \) there exists \( f \in L^2(J) \) such that \( g = Tf \) on \( H \).

(b) If \( g \) is a function of bounded variation on \( H \), there exists \( f \in L^2(J) \)
of bounded variation such that \( g = Tf \) on \( H \).

(c) If \( Dg \) is in \( B.V.(H) \), there exists \( f \in L^2(J) \) such that \( Df \) is in
\( B.V.(J) \) and \( g = Tf \) on \( H \).

In the proof of Theorem 2 we use the following lemma.

**Lemma 2.** If \( g = k * f - f \) on \( J \), where \( Dg \) is in \( B.V.(J) \) and \( f \) is a
measure with support in \( J \) then \( Df \) is in \( B.V.(J) \).

**Proof.** Since \( k * f \) and \( g \) are both functions of bounded variation on \( J \),
\( f \) is a function of bounded variation on \( J \) and therefore on \( R \).
Hence, \( Df \) is a measure, and \( D(k * f) = k * Df \) is in \( B.V.(J) \). Therefore
so is \( Df \), which proves the lemma.

To prove Theorem 2 we first note that \( T \) is a self-adjoint operator
on \( L^2(J) \) and that \( R(T) \) is closed and is the orthogonal complement
of \( N(T) \). Given \( g \in L^2(H) \), we will construct an element of the orthog-
onal complement of \( N(T) \) which equals \( g \) on \( H \).

Since \([k \ast]\) is a completely continuous operator on \( L^2(J) \), \( N(T) \) is
finite-dimensional. Choose \( \{f_i, i = 1, \ldots, n\} \), linearly independent
on \( J \) and spanning \( N(T) \). Then the \( f_i \) are linearly independent on
\( J - H \), as we now show.

Consider any \( f = \sum_{i=1}^{n} c_i f_i, f \neq 0 \) on \( J \), Suppose \( f = 0 \) on \( J - H \). Then
\( f = 0 \) on \( H^c = R - H \). \( f \) is an element of \( N(T) \), i.e., \( k \ast f - f = 0 \) on \( J \).
Lemma 2, where \( g = 0 \), implies that \( Df \) is in \( B.V.(J) \). Therefore \( D^2f \)
is a measure on the interior of \( J \). \( D^2f = 0 \) on \( H^c \). Therefore \( D^2f \) is a
measure on \( R \), with support in \( J \). By Lemma 2, again, \( D^2f \) is in
\( B.V.(J) \). \( D^2f = 0 \) on \( H^c \). Therefore \( D^2f \) is a measure on \( R \). Similarly,
\( D^3f \) is a measure on \( R \) implies \( D^{3+2}f \) is a measure on \( R \), and so by
induction \( D^nf \) is a measure for all \( n \). For any distribution \( h \), \( Dh \) is a
function implies \( h \) is a continuous function, and \( Dh \) is a continuous
function implies $h$ is differentiable and $Dh = h^{(1)}$. Therefore $D^n f = f^{(n)}$, $f \in C^n$, and $f^{(n)} \in L^2(J)$ for all $n$. $f^{(n)}$ is, consequently, an element of the domain of $T$, and $Tf^{(n)} = TD^n f = (k - \delta_0) \ast D^n f = D^n [(k - \delta_0) \ast f]$, which is 0 on the interior of $J$ because $(k - \delta_0) \ast f = 0$ on $J$. Since $f$ is in $C^n$ on all of $R$, so is $D^n [(k - \delta_0) \ast f]$. Therefore $Tf^{(n)} = 0$ on $J$, i.e., $f^{(n)} \in N(T)$ for all $n$. But since $N(T)$ is finite-dimensional, there exists an integer $m$ and constants $b_j, j = 0, \ldots, m$, not all zero, such that $\sum_{j=0}^n b_j f^{(j)} = 0$ on $J$. Consequently $f$ is analytic on the interior of $J$ and cannot be $=0$ on $J - H$. It follows that the $f_i$ are linearly independent on $J - H$ as asserted.

The $f_i$ can be chosen so that their restrictions to $J - H$ form an orthonormal set. For instance, if $g_i = f_i - \sum_{j=1}^m a_{ij} g_j, i = 1, \ldots, n$, is the Gram-Schmidt orthogonalization of the $f_i$ restricted to $J - H$, then the $g_i$ considered as functions on $J$ still span $N(T)$, since each $f_i$ is a linear combination of the $g_i$. We assume that the $f_i$ are so chosen.

Define

$$G(x) = g(x) \quad \text{for } x \in H, \quad G(x) = \sum_{i=1}^n b_i f_i(x) \quad \text{for } x \in J - H,$$

where $b_i = -\int_H f_i(x) g(x) \, dx$. Then $G \in L^2(J)$.

$$\int_J f_i(x) G(x) \, dx = \int_{J-H} f_i(x) \left( \sum_{j=1}^n b_j f_j(x) \right) \, dx + \int_H f_i(x) g(x) \, dx$$

$$= -b_i + b_i = 0,$$

for each $i = 1, \ldots, n$. Therefore $G$ is in the orthogonal complement of $N(T)$ which as noted above is $R(T)$, i.e., there exists $f \in L^2(J)$ such that $G = Tf$ on $J$. Since $g = G = Tf$ on $H$, (a) is proved.

(b) Suppose $g$ is a function of bounded variation on $H$. If $f_i \in N(T)$, i.e., $k \ast f_i = f_i$ on $J$, then $f_i$ is a function of bounded variation. Therefore $G$, as defined above, is a function of bounded variation, and $Tf = k \ast f - f = G$ on $J$ implies $f$ is a function of bounded variation on $J$ and therefore on $R$.

(c) Suppose $Dg$ is in $B.V.(H)$. If $f_i \in N(T)$ then Lemma 2 implies that $Df_i$ is in $B.V.(J)$. Therefore $G$ as defined above is such that $DG$ can fail to be in $B.V.(J)$ only if $G$ has discontinuities at the endpoints, $c$ and $d$, of $H$. To remedy this we will alter the definition of $G$ on $J - H$. Let $h(x)$ be a function on $J - H$ such that

(i) $h$ is in the orthogonal complement of $N(T)$,

(ii) $Dh$ is a function of bounded variation on $J - H$,

(iii) $h(c) = g(c) - \sum_{i=1}^n b_i f_i(c), h(d) = g(d) - \sum_{i=1}^n b_i f_i(d)$.

Define
\[ G(x) = \sum_{i=1}^{n} b_i f_i(x) + h(x) \quad \text{for } x \in J - H, \quad G(x) = g(x) \quad \text{for } x \in H. \]

Then \( DG \) is in \( \text{B.V.}(J) \), and \( G \) is in the orthogonal complement of \( N(T) \). There exists \( f \in L^2(J) \) such that \( G = k \ast f - f \) on \( J \). Lemma 2 implies that \( Df \) is in \( \text{B.V.}(J) \).

We obtain immediately from Theorem 2.

**Corollary 2.** If \( W \) is a function on \([-1, 1]\) such that \( D^n W = k - \delta_0 \) for some positive integer \( n \), where \( k \) is as in Theorem 2, then for every \( g \in L^2(H) \) there exists \( f \in L^2(J) \) such that \( g = W \ast D^n f \) on \( H \).

**References**


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