SOME INEQUALITIES FOR POLYNOMIALS AND THEIR ZEROS

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This note is divided into two parts. In the first part we use some results from the theory of schlicht functions to obtain inequalities involving polynomials and their zeros. Also a new proof is given to a result much used in the theory of polynomials. The second part contains some estimates for the location of the zeros of linear combinations of polynomials. A result due to Biernacki is sharpened and generalized.

I. Lemma. Let

\[ f(z) = \frac{z}{\prod_{k=1}^{n} (1 - \epsilon_k z)^m} \]

Denote \( e = \max_{1 \leq k \leq n} |\epsilon_k| \), then

(a) If \( e \leq 1 \) and \( m \leq 2/n \), the function \( f(z) \) is regular and schlicht in the unit disc.

(b) If \( m = 1 \), \( f(z) \) is regular and schlicht in the disc \( |z| < 1/(n-1)e \).

In general this result cannot be improved.

Proof. For \( |z| = r \), we obtain:

\[ \frac{\partial \arg f(z)}{\partial z} = \text{Re} \left[ z \left( \frac{1}{z} + m \sum_{k=1}^{n} \frac{\epsilon_k}{1 - \epsilon_k z} \right) \right] \]

(1)

\[ = 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \text{Re} \left( \frac{1 + \epsilon_k z}{1 - \epsilon_k z} \right) \]

\[ = 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2} \]

Now \( \partial \arg f(z)/\partial z > 0 \) if \( |\epsilon_k| \leq 1 \) and \( 1 - nm/2 \geq 0 \), for all \( 0 < r < 1 \), which proves (a) since \( \Delta \arg f(z) = 2\pi \), when \( z \) describes a circle of radius \( r \) in the positive direction. Similarly for \( m = 1 \),

(2)

\[ \partial \arg f(z)/\partial z > 0 \quad \text{if} \quad \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2} > n - 2. \]

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\[ \min \frac{1 - |\varepsilon_i|^2 r^2}{|1 - \varepsilon_i z|^2} \leq \frac{1 - \varepsilon R}{1 + \varepsilon R}, \quad \text{for } r \leq R, \]

(2) is satisfied if \( n(1 - \varepsilon R)/(1 + \varepsilon R) > n - 2 \), or \( R < 1/\varepsilon(n - 1) \), which yields (b). The example of Corollary 1(b) provides the final part of the lemma.

**Theorem 1.** Let \( P(z) = a_n z^n + \cdots + a_1 z + a_0 \) have zeros \( \alpha_k, \]
\( k = 1, \ldots, n, \mid \alpha_k \mid \leq 1 \), then \( P(z) = a_n(z - \alpha)^n \), where \( |\alpha(z)| < 1 \), for \( |z| < 1 \), and \( \alpha(z) \) is a regular function in \( |z| > 1 \).

**Proof.** By a reciprocal transformation it is necessary and sufficient to prove

\[ \prod_{k=1}^{n} (1 - z \alpha_k) = (1 - az)^n, \]

where \( |\alpha(z)| < 1 \) for \( |z| < 1 \).

For \( |z| < 1 \) we define \( \alpha(z) \) by equation (3), choosing a single-valued branch of the \( n \)th root. It is easy to see that \( \alpha(z) \) is regular in \( |z| < 1 \).

It remains to show that \( |\alpha(z)| < 1 \) in \( |z| < 1 \).

By the lemma, \( f(z) = z/(1 - az)^2 \) is regular and schlicht in the unit disc. Since it is also normalized by the conditions \( f(0) = 0, f'(0) = 1 \), it follows by a well-known estimate [2], that

\[ \frac{|z|}{(1 + |z|)^2} \leq \frac{|z|}{|1 - az|^2} \leq \frac{|z|}{(1 - |z|)^2}. \]

It follows from (4) that \( |\alpha(z)| < 3 \) in \( |z| < 1 \).

Now since \( 1/(1 - az) \) is regular in the disc \( |z| < 1 \), and the series

\[ \frac{1}{1 - az} = 1 + az + (az)^2 + \cdots \]

converges for \( |z| < 1/3 \), it follows that the series (5) converges for all \( z, |z| < 1 \). Suppose that there is a point \( z_0, |z_0| > 1 \), such that \( |\alpha(z_0)| > 1 \), then by the maximum principle we may assume that \( z_0 \alpha(z_0) > 1 \) and we get a contradiction to the absolute convergence of (5) at \( z_0 \).

**Theorem 2.** Let \( \prod_{k=1}^{n} (1 - \alpha_k z) = (1 - az)^n \), where \( |\alpha_k| \leq 1, |\alpha(z)| \leq 1 \), then

\[ \sum_{k=0}^{\infty} |\beta_k|^2 (2k + 1) \leq 1. \]
where the numbers $\beta_k$ are the coefficients of the expansion in a Taylor series of $\alpha(z)$ in $|z| < 1$.

**Proof.** Since $f(z) = z/(1 - az)^2$ is regular, schlicht and normalized in $|z| < 1$ the same is true for the function $f_2(z) = (f(z^2))^{1/2}$. Define $F(z) = 1/f_2(1/z)$, then $F(z)$ is schlicht in $|z| > 1$ and has the expansion

$$F(z) = z - \frac{1}{z} \beta_0 - \frac{1}{z^2} \beta_1 - \cdots.$$

The theorem follows now by the area theorem for schlicht functions.

The following corollary is deduced easily from the theorems proved.

**Corollary 1.** Let $P(z) = \prod_{k=1}^n (1 - \alpha_k z), |\alpha_k| \leq 1/(n-1)$, then

(a) $(1 + |z|)^2 \geq |P(z)| \geq (1 - |z|)^2$ for $|z| \leq 1$.

(b) $P(z) - zP'(z) \neq 0$ for $|z| < 1$, and this result is the best possible as the example $P(z) = (1 + z/(n-1))^n$ shows.

(c) If $c$ is such that $P(z) - z/c \neq 0$ in $|z| \leq 1$, then

$$(1 + |z|)^2 \geq \left| \frac{P(z) - z}{c} \right| \geq (1 - |z|)^2$$

in $|z| \leq 1$.

(a), (b) follow by the second part of the lemma. (c) follows from the fact that the function $c f(z)/(c - f(z))$ is schlicht and regular in $|z| < 1$, if $f(z) = z/P(z)$.

II. **Lemma.** All the zeros of the polynomial

$$(z + e^{i \theta})^n - 1 - nz - \cdots - \left( \frac{n}{p-1} \right) z^{p-1}$$

are in the disc $|z| \leq p + 1$ for $1 \leq p < n-1; 0 \leq \theta \leq 2\pi$.

**Proof.** We use the inequality

$$(6) \quad 1 + \binom{m + q}{1} (q + 1) + \cdots + \binom{m + q}{q-1} (q + 1)^{q-1} < q^{m+q}.$$

For $m \geq 3, q \geq 2$, (6) was proved by Biernacki [1]. We prove (6) for $m = 2, q \geq 2$ and also indicate the proof for the other cases.

Let $m \geq 2$. It is easy to verify that

$$(7) \quad 1 + \binom{m + q}{1} x + \cdots + \binom{m + q}{q-1} x^{q-1} < \binom{m + q}{q-1} (1 + x)^{q-1}$$

for all $x > 0$. 

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Substituting $x = q + 1$ in (7), one deduces that (6) is true provided

$$w(m, q) = \binom{m + q}{q - 1} (q + 2)^{-1} q^{-m+q} < 1.$$ 

Since $w(m, q)/w(m+1, q) > 1$ for $q > 1$, it is enough to consider the case $m = 2$. In this case after obvious transformations the equivalent to (6) is the inequality $(1 + 2/q)^{t+1} < 1 + (1 + 1/q)^{r+1}(5/2 + 4/q)$.

Since $(1 + 2/q)^{q}$ is increasing, and $(1 + 1/q)^{r+1}$ is decreasing, it is enough to show that $(1 + 2/q)^{q} t < 1 + e(5/2 + 4/q)$.

Set $x = 1 + 2/p$, we get $e^{x^2 - 2ex} < 1 + e/2$, hence

$$x < (1/e)[1 + (2 + e/2)^{1/2}]$$ and $p > 2e[1 + (2 + e/2)^{1/2} - e]^{-1}$.

The remaining cases ($p < 50$) can be verified directly.

The lemma follows now by applying Rouche's theorem, using (6). It follows easily that on the circle $|z| = p + 1$,

$$|z + e^{i\theta}| = \left|1 + n\theta + \cdots + \binom{n}{p - 1} e^{p-1}\right| > p^n.$$ 

It can be shown that the lemma is not true for $p = n - 1$. We prove now

**Theorem 3.** If $P(z) = a_n z^n + \cdots + a_0 \neq 0$ in $|z| < 1$, then the polynomial $P^*(z) = P(z) + \epsilon_n a_n z^n + \epsilon_{n-1} a_{n-1} z^{n-1} + \cdots + \epsilon_{n-p+1} a_{n-p+1} z^{n-p+1} \neq 0$ in $|z| < 1/(p + 1)$, for $|\epsilon_k| \leq 1$, $k = n - p + 1, \ldots, n$; $1 \leq p < n - 1$.

**Proof.** By a result due to Rahman [3], $P^*(z) \neq 0$ in $|z| < 1/t$, where $t$ is the positive root of the equation

$$(t - 1)^n = 1 + n t + \cdots + \binom{n}{p - 1} t^{p-1}.$$ 

By the lemma, with $\theta = \pi$, $k = p + 1$, $t \leq p + 1$, hence $1/t \geq 1/(p + 1)$.

Theorem 3 generalizes a result due to Biernacki [1], obtained for $\epsilon_n = \epsilon_{n-1} = \cdots = \epsilon_{n-p+1} = -1$.

**Bibliography**