SOME INEQUALITIES FOR POLYNOMIALS
AND THEIR ZEROS

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This note is divided into two parts. In the first part we use some results from the theory of schlicht functions to obtain inequalities involving polynomials and their zeros. Also a new proof is given to a result much used in the theory of polynomials. The second part contains some estimates for the location of the zeros of linear combinations of polynomials. A result due to Biernacki is sharpened and generalized.

I. Lemma. Let

\[ f(z) = \frac{z}{\prod_{k=1}^{n} (1 - \epsilon_k z)^{m}}. \]

Denote \( e = \max_{1 \leq k \leq n} |\epsilon_k| \), then

(a) If \( e \leq 1 \) and \( m \leq 2/n \), the function \( f(z) \) is regular and schlicht in the unit disc.

(b) If \( m = 1 \), \( f(z) \) is regular and schlicht in the disc \( |z| < 1/(n - 1)e \).

In general this result cannot be improved.

Proof. For \( |z| = r \), we obtain:

\[
\frac{\partial \arg f(z)}{\partial z} = \text{Re} \left[ z \left( \frac{1}{z} + m \sum_{k=1}^{n} \frac{\epsilon_k}{1 - \epsilon_k z} \right) \right]
\]

(1)

\[
= 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \text{Re} \left( \frac{1 + \epsilon_k z}{1 - \epsilon_k z} \right)
\]

\[
= 1 - \frac{nm}{2} + \frac{m}{2} \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2}.
\]

Now \( \partial \arg f(z)/\partial z > 0 \) if \( |\epsilon_k| \leq 1 \) and \( 1 - nm/2 \geq 0 \), for all \( 0 < r < 1 \), which proves (a) since \( \Delta \arg f(z) = 2\pi \), when \( z \) describes a circle of radius \( r \) in the positive direction. Similarly for \( m = 1 \),

(2) \[ \partial \arg f(z)/\partial z > 0 \quad \text{if} \quad \sum_{k=1}^{n} \frac{1 - |\epsilon_k|^2 r^2}{|1 - \epsilon_k z|^2} > n - 2. \]

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Theorem 1. Let \( P(z) = a_n z^n + \cdots + a_1 z + a_0 \) have zeros \( a_k, k = 1, \ldots, n, |a_k| \leq 1, \) then \( P(z) = a_n (z - \alpha)^n, \) where \( |\alpha(z)| < 1, \) for \( |z| < 1, \) and \( \alpha(z) \) is a regular function in \( |z| > 1. \)

Proof. By a reciprocal transformation it is necessary and sufficient to prove

\[
|z| \frac{|z|}{(1 + |z|)^2} \leq \frac{|z|}{|1 - az|^2} \leq \frac{|z|}{(1 - |z|)^2}.
\]

It follows from (4) that \( |\alpha(z)| < 3 \) in \( |z| < 1. \)

Now since \( 1/(1 - az) \) is regular in the disc \( |z| < 1, \) and the series

\[
\sum_{k=0}^{\infty} |\beta_k| (2k + 1) \leq 1
\]

converges for \( |z| < 1/3, \) it follows that the series (5) converges for all \( z, |z| < 1. \) Suppose that there is a point \( z_0, |z_0| > 1, \) such that \( |\alpha(z_0)| > 1, \) then by the maximum principle we may assume that \( |z_0 \alpha(z_0)| > 1 \) and we get a contradiction to the absolute convergence of (5) at \( z_0. \)

Theorem 2. Let \( \prod_{k=1}^{n} (1 - a_k z) = (1 - az)^n, \) where \( |a_k| \leq 1, |\alpha(z)| \leq 1, \) then

\[
\sum_{k=0}^{n} |\beta_k| (2k + 1) \leq 1
\]
where the numbers \( \beta_k \) are the coefficients of the expansion in a Taylor series of \( \alpha(z) \) in \(|z| < 1\).

**Proof.** Since \( f(z) = z/(1 - az)^2 \) is regular, schlicht and normalized in \(|z| < 1\) the same is true for the function \( f_s(z) = (f(z^2))^{1/2} \). Define \( F(z) = 1/f_s(1/z) \), then \( F(z) \) is schlicht in \(|z| > 1\) and has the expansion

\[
F(z) = z - \frac{1}{z} \beta_0 - \frac{1}{z^2} \beta_1 - \cdots.
\]

The theorem follows now by the area theorem for schlicht functions.

The following corollary is deduced easily from the theorems proved.

**Corollary 1.** Let \( P(z) = \prod_{k=1}^{n} (1 - \alpha_k z) \), \( |\alpha_k| \leq 1/(n - 1) \), then

(a) \( (1 + |z|)^2 \geq |P(z)| \geq (1 - |z|)^2 \) for \(|z| \leq 1\).
(b) \( P(z) - zP'(z) \neq 0 \) for \(|z| < 1\), and this result is the best possible as the example \( P(z) = (1 + z/(n - 1))^n \) shows.
(c) If \( c \) is such that \( P(z) - z/c \neq 0 \) in \(|z| \leq 1\), then

\[
(1 + |z|)^2 \geq \left| P(z) - \frac{z}{c} \right| \geq (1 - |z|)^2
\]

in \(|z| \leq 1\).

(a), (b) follow by the second part of the lemma. (c) follows from the fact that the function \( cf(z)(c - f(z)) \) is schlicht and regular in \(|z| < 1\), if \( f(z) = z/P(z) \).

II. **Lemma.** All the zeros of the polynomial

\[
(z + e^{i\theta})^n - 1 - nz - \cdots - \binom{n}{p-1} z^{p-1}
\]

are in the disc \(|z| \leq p + 1\) for \( 1 \leq p < n - 1 \); \( 0 \leq \theta \leq 2\pi \).

**Proof.** We use the inequality

\[
1 + \binom{m + q}{1} (q + 1) + \cdots + \binom{m + q}{q - 1} (q + 1)^{q-1} < q^{m+q}.
\]

For \( m \geq 3, q \geq 2 \), (6) was proved by Biernacki [1]. We prove (6) for \( m = 2, q \geq 2 \) and also indicate the proof for the other cases.

Let \( m \geq 2 \). It is easy to verify that

\[
1 + \binom{m + q}{1} x + \cdots + \binom{m + q}{q - 1} x^{q-1} < \binom{m + q}{q - 1} (1 + x)^{q-1}
\]

for all \( x > 0 \).
Substituting \( x = q + 1 \) in (7), one deduces that (6) is true provided

\[
\omega(m, q) = \binom{m + q}{q - 1} \frac{1}{q + 2} < 1.
\]

Since \( \omega(m, q)/\omega(m + 1, q) > 1 \) for \( q > 1 \), it is enough to consider the case \( m = 2 \). In this case after obvious transformations the equivalent to (6) is the inequality \( (1 + 2/q)^{q+2} < 1 + (1 + 1/q)^{q+1}(5/2 + 4/q) \).

Since \( (1 + 2/q)^q \) is increasing, and \( (1 + 1/q)^{q+1} \) is decreasing, it is enough to show that \( (1 + 2/q)^{q+2} < 1 + e(5/2 + 4/q) \).

Set \( x = 1 + 2/p \), we get \( e^x < 1 + e/2 \), hence

\[
x < (1/e) \left[ 1 + (2 + e/2)^{1/2} \right] \quad \text{and} \quad p > 2e \left[ 1 + (2 + e/2)^{1/2} - e \right]^{-1}.
\]

The remaining cases \( p > 50 \) can be verified directly.

The lemma follows now by applying Rouche's theorem, using (6).

It follows easily that on the circle \( |z| = \rho + 1 \),

\[
|z + e^\theta|^n > \rho^n > 1 + ns + \cdots + \left( \frac{n}{\rho - 1} \right) s^{\rho - 1}.
\]

It can be shown that the lemma is not true for \( p = n - 1 \). We prove now

**Theorem 3.** If \( P(z) = a_n z^n + \cdots + a_0 \neq 0 \) in \( |z| < 1 \), then the polynomial \( P^*(z) = P(z) + \epsilon_0 a_n z^{n-1} + \cdots + \epsilon_{n-p+1} z^{n-p+1} \neq 0 \) in \( |z| < 1/(\rho + 1) \), for \( |\epsilon_k| \leq 1 \), \( k = n - p + 1, \ldots, n ; 1 \leq \rho < n - 1 \).

**Proof.** By a result due to Rahman \([3]\), \( P^*(z) \neq 0 \) in \( |z| < 1/t \), where \( t \) is the positive root of the equation

\[
(t - 1)^n = 1 + nt + \cdots + \left( \frac{n}{\rho - 1} \right) t^{\rho - 1}.
\]

By the lemma, with \( \theta = \pi \), \( k = \rho + 1 \), \( t \leq \rho + 1 \), hence \( 1/t \geq 1/(\rho + 1) \). Theorem 3 generalizes a result due to Biernacki \([1]\), obtained for \( \epsilon_n = \epsilon_{n-1} = \cdots = \epsilon_{n-p+1} = -1 \).

**Bibliography**


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