A NEW PROOF OF A CONJECTURE OF SCHILD

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1. Introduction. In a paper, published in this Journal [3], one of the authors has introduced and discussed the class of functions $S_p$, having $|z| = 1$ as radius of schlichtness and being of the form $f_p(z) = z - \sum_{n=2}^{N} a_n z^n$, with the $a_n$ real and non-negative for $n = 2, 3, \ldots, N, N \geq 2$. All functions of this class map the unit circle into starlike regions [3, Theorem 3].

Let $d^*$ be the shortest distance from $w = Q$ to $w = f_p(\theta e^{i\theta})$, $0 \leq \theta < 2\pi$, and $d_0$ the shortest distance from $w = 0$ to $w = f_p(r_0 e^{i\theta})$, $0 \leq \theta < 2\pi$, where $r_0$ is the radius of convexity of $w = f_p(z)$. Among other things it was proved [3, Theorem 7] that for all $f_p(z) \in S_p$ we have $d_0/d^* \geq 2/3$.

It was conjectured there that for $f_p(z) \in S_p$ we actually have $d_0/d^* \geq 3/4$, attained by the function $f_p(z) = z - z^2/2 \in S_p$.

The class of functions $S_p$ was discussed further and extended by Z. Lewandowski [1] and the truth of the conjecture $d_0/d^* \geq 3/4$ for all $f_p(z) \in S_p$ was demonstrated by him in a second paper [2].

It is the aim of this short note to give an elementary and simple proof of the conjecture: $d_0/d^* \geq 3/4$ for all $f_p(z) \in S_p$.

2. Proof of the conjecture that $d_0/d^* \geq 3/4$ for all $f_p(z) \in S_p$. The map of $|z| = r$, $0 < r \leq 1$, by any $f_p(z) \in S_p$ will have its closest point from the origin on the positive real axis for $z = r$, since

$$|f_p(z)| = |z - \sum_{n=2}^{N} a_n z^n| \geq |z| - \sum_{n=2}^{N} a_n |z|^n = r - \sum_{n=2}^{N} a_n r^n = f_p(r).$$

We must show, therefore, that

$$d_0/d^* = \frac{r_0 - \sum_{n=2}^{N} a_n r_0^n}{1 - \sum_{n=2}^{N} a_n} \geq 3/4,$$

where $r_0$ is the radius of convexity of $f_p(z)$.

$$d_0/d^* - 3/4 = \frac{r_0 - \sum_{n=2}^{N} a_n r_0^n}{1 - \sum_{n=2}^{N} a_n} - 3/4$$

$$= \frac{r_0 - 3/4 + \sum_{n=2}^{N} a_n (3/4 - r_0^n)}{1 - \sum_{n=2}^{N} a_n}.$$

Since $\{1 - \sum_{n=2}^{N} a_n\} > 0$ [3, Theorem 1], it is sufficient to show that $y = (r_0 - 3/4) + \sum_{n=2}^{N} a_n (3/4 - r_0^n) \geq 0$.

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We recall [3, Theorem 5] that \( r_0 \) is the least positive root of 
\[
\sum_{n=2}^{N} n^2 a_n r_0^{n-1} = 1
\]
and, therefore, 
\[
\sum_{n=2}^{N} n^2 a_n r_0^{n-1} = 1.
\]
The expression for \( y \) can now be written in the form:
\[
y = (r_0 - 3/4) \sum_{n=2}^{N} n^2 a_n r_0^{n-1} + \sum_{n=2}^{N} a_n (3/4 - r_0^n)
\]
\[
= \sum_{n=2}^{N} a_n \{ n r_0 - (3/4)n r_0^{n-1} + 3/4 - r_0^n \}.
\]
The proof will now be completed by showing that
\[
z(n) = \{ n r_0 - (3/4)n r_0^{n-1} + 3/4 - r_0^n \} \geq 0 \quad \text{for } n = 2, 3, \cdots, N.
\]
Clearly, \( z(2) = 3r_0^2 - 3r_0 + 3/4 = 3(r_0 - 1/2)^2 \geq 0 \). For \( n \geq 2 \), we consider
\[
g(n) = z(n + 1) - z(n)
\]
\[
= r_0^{n-1} \{ (n^2 + 2n)r_0 - (7n^2/4 + 3n/2 - 1/4)r_0 + 3n^2/4 \}.
\]
It was shown [4, Lemma 3.4] that \( d_0/d^* > r_0 \), for all functions \( w = z + \sum_{n=2}^{N} a_n z^n \), regular, schlicht and starlike in the unit circle. This result will, therefore, also hold for \( f_p(z) \in S_p \), and since, for this class of functions, \( r_0 \geq 1/2 \) [3, Theorem 5], it is sufficient to prove the conjecture for \( 1/2 \leq r_0 < 3/4 \). It is, therefore, convenient to set \( r_0 = 3/4 - x \), where \( 0 < x \leq 1/4 \), in the expression for \( g(n) \). For any particular \( n \), the coefficient of \( r_0^{-1} \) in \( g(n) \) becomes \( h(x) = (n^2 + 2n)x^2 + (1/4)(7n^2 - 6n - 1)x + 3/16 \). It is clear that \( h(x) > 0 \) for \( n \geq 6 \). Also, the discriminant of \( h(x) \) is \( \Delta = (1/16)(n^4 - 12n^3 + 22n^2 - 12n + 1) \)
\[
= (1/16)(n-1)^2(n^2 - 10n + 1) + 1.
\]
Obviously, \( \Delta < 0 \) for \( n < 10 \), and since \( h(0) > 0 \), \( h(x) > 0 \) for \( n = 2, 3, 4, 5 \) also. Therefore, \( z(n+1) - z(n) > 0 \) for \( n = 2, 3, 4, \cdots \) and since \( z(2) \geq 0 \), the conjecture is proved.

REFERENCES


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