1. Introduction. The purpose of this note is to provide a much simpler and more illuminating proof of the main result of the author's previous paper [7], and also to deduce some interesting corollaries of this result. Our general terminology and notation are the same as [7], to which we refer the reader for any definitions not given here.

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If \((G, R)\) is an unoriented graph, we shall refer to the elements of \(G\) as vertices. If \(x\) and \(y\) are vertices with \(x \sim y\), then the unordered pair \(\{x, y\}\) will be called an edge of \((G, R)\). The notation \((x, y)\) will denote an ordered pair.

We say that a relation \(T\) on \(G\) is an orientation of \((G, R)\) if and only if (i) \((G, T)\) is an oriented graph, and (ii) \(x \sim y\) if and only if \(x \lessdot y\) or \(y \lessdot x\), for all \(x, y \in G\). An unoriented graph which possesses a transitive orientation will be called a comparability graph. Thus, in slightly different language, an unoriented graph \((G, R)\) is a comparability graph if and only if there exists a partial ordering \(\prec\) of the vertices of \(G\), such that \(x \sim y\) if and only if \(x \prec y\) or \(y \prec x\). In this case, \((G, R)\) is called the comparability graph of the partially ordered set \((G, \prec)\).

Recently A. Ghouila-Houri [2] and P. C. Gilmore and A. J. Hoffman [3] have given necessary and sufficient conditions for an unoriented graph to be a comparability graph, but we shall not make use of these conditions here.

2. A preliminary result. We shall first show that the proof that a graph possesses a transitive orientation may always be reduced to the finite case. While such a result is proved in [3], we shall give another proof based on the following well-known theorem of R. Rado [4], [6].

**Theorem 1 (Rado).** Let \(I\) be an arbitrary index set, and let \(\{X_i : i \in I\}\) be a family of finite sets. For each finite subset \(A\) of \(I\), let \(f_A\) be a choice function for the family \(\{X_i : i \in A\}\). Then there exists a choice function \(f\) for \(\{X_i : i \in I\}\) such that, whenever \(A\) is a finite subset of \(I\), there is a finite set \(B, A \subseteq B \subseteq I\), with \(f(i) = f_B(i)\) for all \(i \in A\).
We now prove

**Theorem 2.** If every finite subgraph of an unoriented graph \((G, R)\) admits a transitive orientation, then so does \((G, R)\).

**Proof.** Let \(E\) be the set of all edges of \((G, R)\). If \(e = \{x, y\}\) is a member of \(E\), let \(Z_e\) be the set whose elements are the ordered pairs \((x, y)\) and \((y, x)\): i.e., \(Z_e = \{(x, y), (y, x)\}\). For each finite subset \(A\) of \(E\), let \(f_A\) be a choice function for \(\{Z_e : e \in A\}\) such that the orientation defined by the set of ordered pairs \(\{f_A(e) : e \in A\}\) is a transitive orientation of the subgraph determined by \(A\). Then, by Theorem 1, there exists a choice function \(f\) for \(\{Z_e : e \in E\}\) such that, whenever \(A\) is a finite subset of \(E\), there is a finite set \(B, A \subseteq B \subseteq E\), with \(f(e) = f_B(e)\) for all \(e \in A\). Define \(T = \{f(e) : e \in E\}\). Clearly \(T\) is an orientation of \((G, R)\). To show that \(T\) is transitive, suppose that \(x T y\) and \(y T z\). Yet \(A\) be the set of edges \(\{\{x, y\}, \{y, z\}, \{x, z\}\}\). There is a finite set of edges \(B, \) with \(A \subseteq B\), such that \(f(e) = f_B(e)\) for all \(e \in A\). The orientation determined by \(f_B\) is transitive on the subgraph determined by \(B\). Hence we must have \(f(\{x, z\}) = f_B(\{x, z\}) = (x, z)\), and thus \(x T z\).

3. The comparability graph of a tree. In [7] we said that an unoriented graph \((G, R)\) has the diagonal property if and only if whenever \(x_1, x_2, x_3, x_4\) are distinct vertices of \(G\) with \(x_1 R x_2 R x_3 R x_4\), then \(x_1 R x_2\) or \(x_2 R x_4\). A graph with the diagonal property will be called a **D-graph**. The following theorem was the main result of [7].

**Theorem 3.** An unoriented graph \((G, R)\) is the comparability graph of a tree if and only if \((G, R)\) is a D-graph.

The existence of a transitive orientation on a D-graph \(G\) follows rather simply from the criteria of Ghouila-Houri or Gilmore and Hoffman. However, these theorems do not enable us to conclude that a transitive orientation may be selected for \(G\) so as to make the resulting partially ordered set a tree. This result was proved in [7] by a rather complicated transfinite induction. We now give a much simpler proof, making use of Theorem 2 and the following lemma. Let us say that a vertex \(c\) of a graph \((G, R)\) is a central point if and only if \(c R x\) for all \(x \in G, x \neq c\).

**Lemma.** A finite connected D-graph \((G, R)\) has at least one central point.

**Proof.** Let \(c\) be a vertex of largest degree of \((G, R)\) and let \(a_0, a_1, \ldots, a_m\) be all the vertices joined to \(c\) by edges. Suppose that \(G\)
has vertices other than $c, a_0, a_1, \ldots, a_m$. Then by the connectedness of $(G, R)$ there exists such a vertex $b$ which is joined to one of $a_0, a_1, \ldots, a_m$, say $a_0$. For any $i = 1, 2, \ldots, m$, we have $a_i R c R a_0 R b$, but $c \not{R} b$: hence by the diagonal property we must have $a_i R a_0$. But then $a_0$ is a vertex of greater degree than $c$: a contradiction. Therefore $c, a_0, a_1, \ldots, a_m$ comprise all the vertices of $G$ and $c$ is a central point.

The reader may easily construct examples to show that the above lemma need not hold if $G$ is infinite.

We now prove Theorem 3. We consider only the sufficiency of the diagonal property (a proof of the necessity of this condition is given in [7]). By Theorem 2 we may assume that $G$ is finite. We may also assume that $(G, R)$ is connected; for, if it is not, then it suffices to prove the theorem for each connected component. Our proof is by induction. The number of vertices of $G$ will be denoted by $|G|$. Let us assume that Theorem 3 is true whenever $|G| < n$; and let $(G, R)$ be a connected $D$-graph with $|G| = n$. Let $c$ be a central point of $G$. By our inductive hypothesis, $G - \{c\}$ possesses a transitive orientation $S$, such that $(G - \{c\}, S)$ is a tree. (If $G - \{c\}$ is not connected, we may apply the inductive hypothesis to each connected component.) We adjoin $c$ to $G - \{c\}$ as a “least” element. More precisely, we define a relation $T$ on $G$ by

$$T = S \cup \{(c, x) : x \in G - \{c\}\}.$$  

It is clear that $T$ is a transitive orientation of $(G, R)$. Furthermore, since $G - \{c\}$ is a tree with respect to the orientation $S$, it follows that $(G, T)$ is also a tree.

4. The complement of a $D$-graph. The graph $(G, R)$ will be called the complement of $(G, R)$. The reader may easily verify that a graph $(G, S)$ is the complement of a $D$-graph if and only if whenever $x_1, x_2, x_3, x_4$ are distinct vertices of $G$ with $x_1 S x_2 S x_3 S x_4$, then one of the relations $x_1 S x_3, x_2 S x_4, x_1 S x_4$ must hold. We shall prove the following theorem.

**Theorem 4.** If $(G, R)$ is a $D$-graph, then $(G, R)$ is a comparability graph.

**Proof.** We may assume that $G$ is finite, and let us assume that the theorem holds whenever $|G| < n$. Let $(G, R)$ be a $D$-graph with $|G| = n$ and suppose that $(G, R)$ is connected. Then $(G, R)$ has a central point $c$, which is an isolated point of $(G, R)$. By our inductive hypothesis, $(G - \{c\}, R)$ has a transitive orientation $T$. Since there
exist no edges from $e$ to any vertex of $(G, \bar{R})$, it follows that $T$ also defines a transitive orientation of $(G, \bar{R})$. Thus the theorem is true whenever $(G, R)$ is connected.

If $(G, R)$ is not connected, let $(G_1, R_1), \ldots, (G_m, R_m)$ be its connected components. By the preceding argument each graph $(G_i, \bar{R}_i)$, for $i = 1, \ldots, m$, possesses a transitive orientation. But in the graph $(G, \bar{R})$ every vertex of $G_i$ is joined by an edge to every vertex of $G_j$, for $i \neq j$. Hence it is clear that the existence of a transitive orientation on each $(G_i, \bar{R}_i)$ implies that $(G, \bar{R})$ may be transitively oriented.

The concept of the dimension of a partially ordered set has been introduced by Dushnik and Miller [1]. We refer the reader to [1] and to Ore [5] for the relevant definitions. We now have the following corollary of Theorem 4.

**Theorem 5.** The dimension of a tree is $\leq 2$.

**Proof.** If $(G, R)$ is the comparability graph of the partially ordered set $(G, \prec)$, and if $(G, R)$ is also a comparability graph, then Dushnik and Miller [1, Theorem 3.61] have shown that the dimension of the partially ordered set $(G, \prec)$ is $\leq 2$. Theorem 5 thus follows immediately from Theorem 4.

**References**


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