ON SUBSPACES OF THE SPACE \((m)\)

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In this note, \(m\), \(c\), and \(c_0\) denote the Banach spaces of bounded, convergent, and null sequences, with the norm

\[
\|x\| = \sup_n |s_n| \quad (x = \{s_n\} \subseteq m).
\]

**Theorem.** If \(A\) is a given matrix summability method, \(X_0\) the subspace of \(m\) consisting of all bounded sequences summed to zero by \(A\), and \(c_0\) a proper subspace of \(X_0\), then there exists no bounded projection of \(X_0\) onto \(c_0\).

**Corollary.** If \(X\) is the space of all bounded sequences summed by \(A\), and \(c\) a proper subspace of \(X\), then there exists no bounded projection of \(X\) onto \(c\).

The corollary was conjectured by Wilansky [6, p. 250]; relevant known results are that \(X_0\) and \(X\) are nonseparable ([2, 3.6.2] and [1, pp. 97–99]), that there exists a bounded projection onto \(c\) (or \(c_0\)) of any separable subspace of \(m\) containing \(c\) (or \(c_0\)) ([3, Theorem 2.2] and [5, Theorem 5]), and that there exists no bounded projection of \(m\) onto \(c\) or \(c_0\) ([4, p. 539] and [5, p. 945]).

To prove the theorem, observe that by [2, 3.6], there exists a closed subspace \(Y_0\) of \(X_0\), a bounded linear operator \(T\) from \(m\) onto \(Y_0\), and a strictly increasing sequence \(\{m_p\}\) of positive integers, such that for \(p = 1, 2, \ldots\),

\[
(1) \quad t_{m_p} = u_p \quad \text{for all } \{u_n\} \subseteq m, \text{ where } T(\{u_n\}) = \{t_n\}.
\]

The construction given in [2, 3.6] for this operator \(T\) ensures also that \(T(c_0) \subseteq c_0\).

Now suppose if possible that there exists a bounded projection \(P\) of \(X_0\) onto \(c_0\). Let \(Q_1: Y_0 \to c_0\) be the restriction of \(P\) to \(Y_0\), and define the bounded linear operator \(T_1 = Q_1 T: m \to c_0\). Since \(T(c_0) \subseteq c_0 \cap Y_0\),

\[
(2) \quad T_1(\{u_n\}) = T(\{u_n\}) \quad \text{when } \{u_n\} \subseteq c_0.
\]

Define a bounded linear operator \(R: c_0 \to c_0\) by

\[
(3) \quad v_p = s_{m_p} \quad \text{for all } \{s_n\} \subseteq c_0, \text{ where } R(\{s_n\}) = \{v_n\},
\]

for \(p = 1, 2, \ldots\). Let \(Q_2\) be the restriction of \(R\) to the range of \(T_1\),

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and define $T_2 = Q_2T_1: m \to c_0$. By (1), (2), and (3), if $\{u_n\} \subseteq c_0$, and $T_2(\{u_n\}) = \{v_n\}$, then $u_p = v_p$ for $p = 1, 2, \cdots$; thus $T_2$ is a bounded projection of $m$ onto $c_0$. This contradicts the result of Sobczyk [5, p. 945], and the theorem is proved.

For the corollary, let $(a_{n,k})$ be the matrix of the method $A$, and define a method $B$ by the matrix $(b_{n,k})$, where

$$b_{n,k} = a_{n,k} - \lim_{n \to \infty} a_{n,k};$$

let

$$\lambda(B) = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_{n,k},$$

and let $X'_0$ be the space of bounded sequences summed to zero by $B$. That $c_0$ is a proper subspace of $X'_0$ follows from the hypothesis that $c$ is a proper subspace of $X$ when $\lambda(B) \neq 0$, and follows from [1, p. 97] when $\lambda(B) = 0$. By the theorem, and since $X'_0 \subseteq X$, there exists no bounded projection of $X$ onto $c_0$, but by [5, p. 938], there exist bounded projections of $c$ onto $c_0$. The corollary is thus proved.

References

5. A. Sobczyk, Projection of the space $(m)$ on its subspace $(c_0)$, Bull. Amer. Math. Soc. 47 (1941), 938–947.

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