1. Introduction. It is well known that there exists a homeomorphism between two Q-spaces (= realcompact spaces) X and Y if and only if there exists an isomorphism between $C(X)$ and $C(Y)$, their rings of real-valued, continuous functions. This suggests the problem of finding algebraic conditions relating $C(X)$ and $C(Y)$ which are both necessary and sufficient for embedding Y in X. Our investigations in this direction have led to the consideration of three types of ring homomorphisms. Before we define these, let us recall that an ideal $M$ of a ring $A$ is a real ideal if $A/M$ is isomorphic to $R$, the field of real numbers. We shall refer to the intersection of a collection of real ideals as a $\delta$-real ideal, and we say that a subring $B$ of a ring $A$ is $\delta$-dense in $A$ if for every pair $M_1$ and $M_2$ of $\delta$-real ideals of $A$ with $M_1 - M_2 \neq \emptyset$, $M_1 - M_2$ contains an element of $B$.

**Definition (1.1).** A homomorphism from a ring $A$ into a ring $B$ is a $\delta$-homomorphism if it is nontrivial and the image of $A$ is $\delta$-dense in $B$.

We shall refer to a set of elements of a ring as subreal if it is contained in a real ideal of the ring.

**Definition (1.2).** A $\delta$-homomorphism is a $\delta F$-homomorphism if the image of every real ideal containing the kernel is subreal.

**Definition (1.3).** A $\delta$-homomorphism from a ring $A$ into a ring $B$ is a $\delta G$-homomorphism if for every real ideal $M$ of $A$ whose image is subreal, there exists an element $a \in M$ such that the image of every real ideal not containing $a$ is subreal.

The four main results of this paper are given in Theorems (2.3), (2.5), (2.6) and (2.7). From these it follows that a Q-space $Y$ can be embedded in a Q-space $X$ if and only if there exists a $\delta$-homomorphism from $C(X)$ into $C(Y)$ and that $Y$ can be embedded in $X$ as a closed (open, dense) subset if and only if there exists a $\delta F$-homomorphism ($\delta G$-homomorphism, $\delta$-isomorphism) from $C(X)$ into $C(Y)$.

2. The embedding theorems. It will be assumed that all topological spaces discussed here are completely regular and Hausdorff.
Theorem (2.1). Let $X$ be a $Q$-space, $Y$ an arbitrary space, and $\phi$ a homomorphism from $C(X)$ into $C(Y)$. Then $\phi$ has the property that

(2.1.1) for every $g \in C(Y)$ and $y \in Z(g)$, there exists an $f \in C(X)$ such that $y \in Z(\phi f)$ and $Z(g) \subseteq Z(\phi f)$

if and only if there is a homeomorphism $h$ from $Y$ into $X$ such that $\phi f = f \circ h$ for all $f \in C(X)$.

Proof. (Sufficiency). Suppose $\phi f = f \circ h$ for all $f \in C(X)$ and that $y \in Z(g)$ where $g \in C(Y)$. Then $h(y) \in Cl_x(h[Z(g)])$ and there exists a function $f \in C(X)$ which vanishes on $Cl_x(h[Z(g)])$ but not at $h(y)$. It follows that $y \in Z(\phi f)$ and $Z(g) \subseteq Z(\phi f)$.

(Necessity). Now let $\phi$ be a homomorphism from $C(X)$ into $C(Y)$ satisfying (2.1.1). We define a mapping $h$ from $Y$ into $X$ as follows. Let $y \in Y$ be given. Then $\phi_y: C(X) \to \mathbb{R}$ is a homomorphism where $\phi_y$ is given by $\phi_y f = (\phi f)(y)$. Moreover, since $y \in Z(1)$, (2.1.1) guarantees the existence of an element $f \in C(X)$ such that $y \in Z(\phi f)$. Hence $\phi_y f \neq 0$, i.e., $\phi_y$ is a nontrivial homomorphism into $\mathbb{R}$. Since $X$ is a $Q$-space, there exists a unique $x \in X$ such that for each $f \in C(X)$, $\phi_y f = f(x)$. We define $h(y) = x$. Then for any $f \in C(X)$, $\phi f = f \circ h$ and it remains for us to show that $h$ is a homeomorphism.

For any $Z$-set $Z(f)$ of $X$, we have $h[Z(f)] = (f \circ h)^{-1}(0) = Z(\phi f)$ and since the $Z$-sets of $X$ form a basis for the closed sets, we conclude that $h$ is continuous.

Now if $y_1$ and $y_2$ are two distinct elements of $Y$, there is a function $g \in C(Y)$ such that $g(y_1) = 0$ and $g(y_2) = 1$. Then by (2.1.1) there is an $f \in C(X)$ such that $y_1 \in Z(\phi f)$ and $y_2 \in Z(\phi f)$. Hence $f(h(y_1)) = \phi_y f(y_1) \neq 0$ while $f(h(y_2)) = \phi_y f(y_2) = 0$. Therefore $h(y_1) \neq h(y_2)$ and $h$ is one-to-one.

To conclude the proof, we need only show $h^{-1}$ is a continuous mapping from $h[Y]$ onto $Y$. Choose $g \in C(Y)$ and $y \in Z(g)$. Let $f$ denote the function whose existence is guaranteed by (2.1.1). It follows that $h(y) \in h[Y] \cap [X - Z(f)] \subseteq h[Y - Z(g)]$. Hence $h$ is an open mapping which implies $h^{-1}$ is continuous.

If $X$ is any topological space and $F$ is a closed subset of $X$, then $\{ f \in C(X) : F \subseteq Z(f) \}$, which we will denote hereafter by $M_F$, is an ideal of $C(X)$. Moreover, $M_F = \cap \{ M_x : x \in F \}$ and hence is a $\delta$-real ideal. In the case of $Q$-spaces, the converse is also true. That is, if $M$ is a $\delta$-real ideal of $C(X)$ where $X$ is a $Q$-space, there is a unique closed subset $F \subseteq X$ such that $M = M_F$. In fact, $X$ is a $Q$-space if

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* $Z(g)$ denotes the set of points on which $g$ vanishes and is referred to as a $Z$-set.

* For any real number $k$, $k$ denotes the function which maps every point of the space into $k$. 

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and only if every $\delta$-real ideal of $C(X)$ is of the form $M_F$ for some closed subset $F$ of $X$. We shall use these facts without explicit mention.

**THEOREM (2.2).** Let $X$ be an arbitrary topological space and $Y$ a $Q$-space. The following statements concerning a homomorphism $\phi$ from $C(X)$ into $C(Y)$ are equivalent.

1. $(2.2.1)$ For every $g \in C(Y)$ and $y \in Z(g)$, there exists an $f \in C(X)$ such that $y \in Z(\phi f)$ and $Z(g) \subseteq Z(\phi f)$.
2. $(2.2.2)$ $\phi$ is a $\delta$-homomorphism.
3. $(2.2.3)$ The image of $C(X)$ separates points and closed sets and is contained in no $\delta$-real ideal of $C(Y)$.
4. $(2.2.4)$ $\phi$ is the identity mapping on constant functions and the image of $C(X)$ separates points and closed sets.$^4$

**Proof.** $(2.2.1) \Rightarrow (2.2.2)$. Let $M_F$ and $M_H$ be two $\delta$-real ideals of $C(Y)$ with $M_F - M_H \neq \emptyset$. Then $H \subseteq F$ and there is an element $y \in H - F$ and a function $g \in C(Y)$ which vanishes on $F$ but not at $y$. According to $(2.2.1)$, there exists an $f \in C(X)$ such that $y \in Z(\phi f)$ and $Z(g) \subseteq Z(\phi f)$. Hence $\phi f \in M_F - M_H$.

$(2.2.2) \Rightarrow (2.2.3)$. Let $M_F$ be any $\delta$-real ideal of $C(Y)$. If $F = Y$, then $M_F = (0)$ and since $\phi$ is nontrivial, the image of $C(X)$ cannot be contained in $M_F$. On the other hand, if $F \neq Y$, choose $y \in F$. Then $M_Y - M_F \neq \emptyset$ and hence must contain an element of the image of $C(X)$. In either case, $\phi[C(X)]$ is not contained in $M_F$. Now then, if $y \in F$, $M_F - M_Y \neq \emptyset$ and $\phi f \in M_F - M_Y$ for some $f \in C(X)$. Hence $\phi f(y) \notin \text{Cl}(\phi f[F])$, i.e., $\phi[C(X)]$ separates points and closed sets.

$(2.2.3) \Rightarrow (2.2.4)$. Consider $\phi 1$ and suppose there exists a point $y \in Y$ such that $\phi 1(y) = 0$. Then for any $f \in C(X)$, $\phi f(y) = [\phi f(y)][\phi 1(y)] = 0$ which implies $\phi[C(X)] \subseteq M_y$, a contradiction since the image of $C(X)$ is contained in no $\delta$-real ideal. Hence $\phi 1(y) = 1$ for every $y \in Y$, i.e., $\phi 1 = 1$. By induction, it follows that $\phi n = n$ for every positive integer $n$ and from this it follows that $\phi r = r$ for every rational number $r$. Using the fact that the rationals are dense in $R$ and that $\phi$ is also a lattice homomorphism, it can be shown that $\phi k = k$ for every real number $k$.

$(2.2.4) \Rightarrow (2.2.1)$. Suppose $g \in C(Y)$ and $y \in Z(g)$. Then since $\phi[C(X)]$ separates points and closed sets, there is a function $f \in C(X)$ such that $k = \phi f(y) \notin \text{Cl}(\phi f[Z(g)])$. Let $f_1 = f - k$. Since $\phi$ is the identity mapping on constant functions, $\phi f_1(y) = 0$ and $\phi f_1(y) \notin \text{Cl}(\phi f_1[Z(g)])$. Hence there exists a positive number $\epsilon$ such that

$^4$ S. Mrówka pointed out that $(2.2.4)$ is equivalent to $(2.2.1)$. 

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\[ |\phi f(x)| > \varepsilon \quad \text{for} \quad \varepsilon \in \mathbb{R}. \]  
Now let \( f_{\varepsilon} = \varepsilon - |f_1| \). Since \( |f| \leq |f_1| \), \( \phi f_{\varepsilon}(y) \neq 0 \) and \( Z(g) \subset Z(\phi f_{\varepsilon}) \).

From Theorems (2.1) and (2.2), we get

**Theorem (2.3).** Let \( X \) and \( Y \) be \( Q \)-spaces. A homomorphism from \( C(X) \) into \( C(Y) \) is a \( \delta \)-homomorphism if and only if there exists a homeomorphism \( h \) from \( Y \) into \( X \) such that \( \phi f = f \circ h \) for all \( f \in C(X) \).

The following lemma is needed.

**Lemma (2.4).** Let \( X \) and \( Y \) be topological spaces and \( h \) a continuous function from \( Y \) into \( X \). Define a homomorphism \( \phi \) from \( C(X) \) into \( C(Y) \) by \( \phi f = f \circ h \). Then for real ideals \( M_x \) and \( M_y \) of \( C(X) \) and \( C(Y) \), respectively, \( \phi[M_x] \subset M_y \) if and only if \( h(y) = x \).

**Proof** Suppose \( h(y) = x \). Then for \( f \in M_x \), \( \phi f(y) = f(h(y)) = f(x) = 0 \). Thus \( \phi[M_x] \subset M_y \). Conversely, suppose \( \phi[M_x] \subset M_y \) and let \( f \in C(X) \). Then \( f \circ h \in M_y \) and \( f \circ h \in M_x \). Hence \( f(h(y)) = \phi f(y) = f(x) \) for every \( f \in C(X) \) which implies that \( h(y) = x \).

**Theorem (2.5).** Let \( X \) and \( Y \) be \( Q \)-spaces. A homomorphism from \( C(X) \) into \( C(Y) \) is a \( \delta F \)-homomorphism if and only if there exists a homeomorphism \( h \) from \( Y \) into \( X \) such that \( \phi f = f \circ h \) for all \( f \in C(X) \), and \( h[Y] \) is a closed subset of \( X \).

**Proof (Sufficiency).** Let \( M \) be a real ideal of \( C(X) \) which contains the kernel, \( K(\phi) \), of \( \phi \). Then \( M = M_x \) for some \( x \in X \). Moreover, since \( h[Y] \) is closed, \( x \in h[Y] \) since otherwise there would exist a function \( f \in C(X) \) vanishing on \( h[Y] \) but not at \( x \). This would imply \( f \in K(\phi) - M_x \), a contradiction. Then \( x = h(y) \) for some \( y \in Y \) and by the previous lemma, \( \phi[M_x] \subset M_y \), that is, the image of \( M \) is subreal in \( C(Y) \).

**(Necessity)** Now suppose \( \phi \) is a \( \delta F \)-homomorphism. By Theorem (2.3), there is a homeomorphism \( h \) from \( Y \) into \( X \) such that \( \phi f = f \circ h \) for all \( f \in C(X) \). It remains to show that \( h[Y] \) is a closed subset of \( X \). Choose \( x \in Cl(h[Y]) = F \). Then \( K(\phi) = M_F \) and \( M_F \subset M_x \). Therefore there is a real ideal \( M_y \) of \( C(Y) \) such that \( \phi[M_x] \subset M_y \), which, by Lemma (2.4), implies \( x = h(y) \). Thus \( x \in h[Y] \) and we conclude that \( h[Y] = Cl(h[Y]) \).

**Theorem (2.6).** Let \( X \) and \( Y \) be \( Q \)-spaces. A homomorphism \( \phi \) from \( C(X) \) into \( C(Y) \) is a \( \delta G \)-homomorphism if and only if there exists a homeomorphism \( h \) from \( Y \) into \( X \) such that \( \phi f = f \circ h \) for all \( f \in C(X) \), and \( h[Y] \) is an open subset of \( X \).

**Proof (Sufficiency).** Let \( M_x \) be a real ideal of \( C(X) \) whose image is subreal in \( C(Y) \). Hence for some \( y \in Y \), \( \phi[M_x] \subset M_y \). Again using
Lemma (2.4), we have \( h(y) = x \). Then \( x \in X - h[Y] \) which is a closed subset of \( X \) and there is an \( f \in C(X) \) which vanishes on \( X - h[Y] \) but not at \( x \). Then \( f \in M_s \) and it follows that the image of every real ideal not containing \( f \) is subreal.

(Necessity) Now suppose \( \phi \) is a \( \delta G \)-homomorphism and \( x \in h[Y] \). Then \( x = h(y) \) for some \( y \in Y \) and \( \phi[M_s] \subset M_p \). Therefore there exists a function \( f \in C(X) \) such that \( f \in M_s \) and the image of every real ideal not containing \( f \) is subreal. From this it follows that \( x \in [X - Z(f)] \subset h[Y] \) which in turn implies that \( h[Y] \) is open.

Finally, we note that if \( h \) is a continuous function from \( Y \) into \( X \), the homomorphism \( \phi \) given by \( \phi f = f \circ h \) is an isomorphism if and only if \( h[Y] \) is dense in \( X \). This, in conjunction with Theorem (2.3), gives:

**Theorem (2.7).** Let \( X \) and \( Y \) be \( Q \)-spaces. A homomorphism \( \phi \) from \( C(X) \) into \( C(Y) \) is a \( \delta \)-isomorphism if and only if there exists a homeomorphism \( h \) from \( Y \) into \( X \) such that \( \phi f = f \circ h \) for all \( f \in C(X) \) and \( h[Y] \) is a dense subset of \( X \).

**Bibliography**


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