TAME ARCS ON DISKS

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It is the goal of this note to show that each disk in \( E^3 \) contains a tame arc which intersects the boundary of \( D \). In [1] Bing shows that each disk in \( E^3 \) contains many tame arcs. The reason that the arguments given in [1] do not show that each disk contains a tame arc intersecting the boundary is that a disk in \( E^3 \) need not lie on a closed surface in \( E^3 \) [7]. This difficulty can be overcome using Bing's improvement of the "side approximation theorem" [2] and a theorem of Hempel [6].

Suppose that \( D \) is a disk in \( E^3 \).

**Lemma.** If \( D \) lies on a 2-sphere in \( E^3 \) then \( D \) contains a tame arc which intersects both \( \text{Int} \ D \) and \( \text{Bd} \ D \).

**Proof.** Let \( S \) be a 2-sphere in \( E^3 \) containing \( D \). It follows from [1] that for each positive number \( \epsilon \) there exists a tame 2-sphere \( S' \) such that (i) \( S \cap S' \) contains a tame Sierpiński curve \( X \) and, (ii) each component of \( S-X \) is of diameter less than \( \epsilon \).

Now if \( \epsilon \) is chosen less than \( \min \{ \text{diam} \ D, \text{diam} \ (S-D) \} \) then \( X \) must intersect both \( D \) and \( S-D \), and hence \( \text{Bd} \ D \). It follows that \( D \) contains a tame arc which intersects both \( \text{Int} \ D \) and \( \text{Bd} \ D \). This establishes the lemma.

**Theorem.** \( D \) contains a tame arc which intersects both \( \text{Int} \ D \) and \( \text{Bd} \ D \).

**Proof.** Let \( J_1, J_2, \ldots \) be a sequence of tame simple closed curves on \( D \) such that if \( D_1, D_2, \ldots \) are the disks on \( D \) bounded, respectively, by \( J_1, J_2, \ldots \) then \( D_i \subset \text{Int} \ D_{i+1} \) and \( \cup D_i = \text{Int} \ D \). The existence of these tame simple closed curves follows from [1]. It follows from a theorem of Hempel [6] that for each \( i \), \( D_i \) lies on a closed surface in \( E^3 \). This is because \( D_i \) is interior to the larger disk \( D_{i+1} \). Now, using this fact and repeatedly applying the results of [2] and the techniques of [1], there exist tame disks \( D_i', D_i', \ldots \) such that

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(a) $\text{Bd } D'_i = J_i$,

(b) $D_i \cap D'_i$ is a Sierpiński curve,

(c) $D'_i \subset \text{Int } D_{i+1}$, and

(d) $\text{Cl}[UD'_i]$ is a disk bounded by $\text{Bd } D$.

The procedure for obtaining these disks is, roughly, as follows: a dense, null sequence of disks is removed from the interior of $D_1$ and each of these disks is replaced by a tame disk. The resulting tame disk is $D'_1$. Then, disks are removed from the annulus on $D$ bounded by $J_1$ and $J_2$ and are replaced by tame disks to obtain $D'_i$. This process is continued. Care is exercised in replacing disks with tame disks so that each of $D'_i$ and $\text{Cl}[UD'_i]$ is a disk. It follows from a theorem of Gillman [4] that the disks which are removed at the $i$th stage need not intersect $J_i$. For more details on this replacing process the reader is referred to [1].

Now let $D'$ denote $\text{Cl}[UD'_i]$. Notice that $D \cap D'$ is a Sierpiński curve which contains $\text{Cl}[\bigcup J_i]$. Now $D'$ is locally tame at each point of $\text{Int } D'$ and it follows from [3] that there is no loss in generality in assuming that $D'$ is locally polyhedral at each point of $\text{Int } D'$. It follows from [5] that $D'$ lies on a 2-sphere in $E^3$.

Now by the lemma there exists a tame arc $\alpha$ on $D'$ which intersects both $\text{Int } D'$ and $\text{Bd } D'$. Without loss of generality we may assume that $\alpha \cap \text{Bd } D = \{ P \}$. Let $\beta$ be an arc in $D \cap D'$ having $P$ for one end-point and such that $\beta - \{ P \} \subset \text{Int } D$.

Let $K$ be a subdisk of $D'$ such that (i) $\alpha \cup \beta \subset K$, (ii) $K \cap \text{Bd } D' = \{ P \}$ and, (iii) $K$ is locally polyhedral except at $P$. Then there is a 2-sphere $S$ in $E^3$ such that $K \subset S$ and $S$ is locally polyhedral except at $P$. But $P$ lies on the tame arc $\alpha$ and it follows from [8] that $S$ is a tame 2-sphere. Thus the arc $\beta$ is tame and satisfies the conclusion of the theorem. This establishes the theorem.

Notice that the arguments given actually show that the set of points of $\text{Bd } D$ which are accessible by tame arcs from $\text{Int } D$ is dense in $\text{Bd } D$.

Corollary. If $\epsilon > 0$ then there is a triangulation $T$ of $D$ such that (i) mesh $T < \epsilon$ and (ii) if $\sigma$ is a wild 1-simplex of $T$ then $\sigma \subset \text{Bd } D$.

References

3. ———, Locally tame sets are tame, Ann. of Math. (2) 59 (1954), 145–158.


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**SPECIAL \( n \)-MANIFOLDS WITH BOUNDARY**

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By a K-R manifold we mean an \( n \)-manifold with boundary \( M^n \) such that \( \text{Int } M^n = E^n \) and \( \text{Bd } M^n = E^{n-1} \); \( \text{Int } M^n \) and \( \text{Bd } M^n \) are the interior and boundary of \( M^n \) respectively. Both Cantrell [2] and Doyle [3] have shown that for \( n \neq 3 \), each K-R manifold is the product \( E^{n-1} \times [0, 1) \). But for \( n = 3 \) there are infinitely many K-R manifolds which are topologically distinct as pointed out in [4] and [5]. We will investigate certain properties of these manifolds with boundary.

**Lemma 0.** Let \( M^n \) be a K-R manifold. Then \( M^n \) is the product \( E^{n-1} \times [0, 1) \) if each compact set in \( M^n \) lies in a closed \( n \)-cell in \( M^n \).

**Proof.** The proof is simple in that \( M^n \) can be represented as a union of closed \( n \)-cells \( \bigcup C_i \) where \( C_i \cap \text{Bd } M^n \) is an \((n-1)\)-cell \( D_i \) nicely imbedded in \( \text{Bd } C_i \) and \( \text{Bd } M^n \), \( D_i \subset \text{Int } D_{i+1} \) and \( C_i - D_i \subset \text{Int } C_{i+1} \), while \( [C_{i+1} - C_i] \) is an \( n \)-cell. One can then construct a homeomorphism of \( M^n \) onto a copy of \( E^{n-1} \times [0, 1) \).

**Lemma 1.** Let \( M^n \) be an \( n \)-manifold with boundary. If \( C \) is a compact set in \( M^n \) such that \( C \cap \text{Bd } M^n \) lies in an open \((n-1)\)-cell in \( \text{Bd } M^n \), then there is a pseudo-isotopy \( h_t \) of \( M^n \) onto \( M^n \) such that \( h_t(C) \subset F \cup C' \), where \( F \) is a fiber in a collar about \( \text{Bd } M^n \), and \( C' \) is a compact set in \( \text{Int } M^n \).

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