A DIRECT PROOF OF TWO THEOREMS ON TWO-LINE PARTITIONS

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1. Statement of results. B. Gordon has recently proved the two following theorems [1]. Let \( n \) be a fixed positive integer and suppose that

\[
(1.1) \quad n = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j, \quad s \leq r,
\]

a "two-line" partition of \( n \) in which the \( a_i \) and \( b_j \) are positive integers satisfying

\[
\begin{align*}
& a_i > a_{i+1}, \quad 1 \leq i \leq r - 1; \quad b_j > b_{j+1}, \quad 1 \leq j \leq s - 1; \\
& a_j \geq b_j, \quad 1 \leq j \leq s,
\end{align*}
\]

including the case in which the \( b_j \) are absent from (1.1). Then we have

**Theorem 1.** The number of solutions of (1.1) and (1.2) is \( p(n) \), the number of unrestricted partitions of \( n \).

**Theorem 2.** The number of such solutions in odd integers is \( p\left(\lfloor n/2 \rfloor\right) \), where \( p(0) = 1 \).

In Gordon's paper, these results are obtained indirectly with the aid of generating functions and a combinatorial lemma which is proved by induction, and on p. 873 the question of finding a proof by one-to-one correspondence is raised. In the present paper we obtain such a correspondence in a reasonably simple and satisfactory manner.\(^1\) It will be seen that our work closely parallels that of [1], and at the end of §3, we outline an alternative method based on a simpler proof of the above-mentioned lemma. Otherwise, this paper is self-contained except for a few elementary concepts of partition theory explained in [2, Chapter 19].

1a. Notation and conventions. Unless otherwise indicated, small Roman and Greek letters with or without subscripts will denote positive integers and capital letters will denote partitions. We denote a partition \( A \) of \( n \) by

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\(^1\) S. A. Burr has announced a direct proof of Gordon's theorems (Notices Amer. Math. Soc. 10 (1963), 367). To our knowledge, however, his paper has not yet appeared.
A = (w₁, \cdots, w₉) = (wᵢ | 1 \leq i \leq q),

where

\begin{equation}
(1.3) \quad n = \sum_{p=1}^{q} w_p; \quad w_p \geq w_{p+1}, \quad 1 \leq p \leq q - 1.
\end{equation}

When necessary, we write \( A_n \) to emphasize that \( A \) is a partition of \( n \).

In numerical examples we omit commas between summands of partitions and terms of sequences and write, for example, \( A = (321) \) or \( (3 \ 2 \ 1) \) instead of \( (3, 2, 1) \).

All summation indices are subscripts whose range, such as that of \( p \) in (1.3), will be omitted when the index assumes all values allowed by the context. We make the usual convention about vacuous conditions, so that, in particular, empty sums are equal to zero. This means that there is just one partition of 0, the empty partition.

For ease of comparison we shall use the notation of [1] wherever it is relevant, including the term \( \delta \)-partition for a solution of (1.1) and (1.2). As is pointed out in [1], a \( \delta \)-partition may be considered as a partition with two rows of summands, the \( a_i \) forming the top row. The term "partition," otherwise unqualified, will denote an ordinary partition normalized by (1.3).

2. A pictorial argument and reduction of the problem. Let \( P = (x₁, \cdots, x_m) \) be a partition of \( n \), and let \( (y₁, \cdots, yₙ) \) be the conjugate to \( P \) in the graphical sense. Here we specify that \( P \) is given by the horizontal reading of its graph, as in Figure 1. We now draw

a line \( \lambda \) in the position shown in the figure and obtain two partitions \( Q₁ \) and \( Q₂ \) with distinct summands, represented by the sets of horizontal or vertical lines connecting the nodes. More precisely, we have

\begin{equation}
(2.1) \quad Q₁ = (dᵢ | 1 \leq i \leq k), \quad Q₂ = (eᵢ | 1 \leq i \leq \kappa),
\end{equation}

where
\[(2.2) \quad d_f = y_f - f, \quad e_g = x_g - (g - 1), \quad \kappa - 1 \leq h \leq \kappa,\]

in which \(\kappa^2\) is the number of nodes in the Durfee square of \(P\). (\(h = 0\) and \(Q_1\) is absent only if \(P = (e_i)\).) In Figure 1 we have \(P = (7754311), \quad Q_1 = (7631), \quad Q_2 = (632), \quad \kappa = 4, \quad h = 3, \) and the Durfee square is enclosed by the light dashed line.

Hence there is a 1-1 correspondence, given by (2.1), between the ordinary partitions \(P\) and the ordered pairs \(D = [Q_1, Q_2]\), in which, for arbitrary \(\kappa, Q_1\) and \(Q_2\) are any partitions with \(h\) and \(\kappa\) distinct summands respectively, where \(\kappa = h\) or \(h+1\), and \(h \geq 0\). We call \(D\) a diagonal partition whose summands are defined to be the numbers \(d_f\) and \(e_g\) of (2.1). If \(D\) corresponds to \(P\) and \(P = P_\alpha\), then we write \(D = D_\alpha\). Here and in the sequel all correspondences are 1-1.

We now show that the number of diagonal partitions \(D_\alpha\) with odd summands is \(p([n/2])\), a fact needed for the proof of Theorem 2. Let \(D_\alpha = [Q_1, Q_2]\) be a diagonal partition of \(n\) whose summands \(d_f\) and \(e_g\) are all odd. Put

\[(2.3) \quad \begin{align*}
    d_f &= 2d_f - 1 & (1 \leq f \leq h), \\
    e_g &= 2e_g - 1 & (1 \leq g \leq \kappa).
\end{align*}\]

It is clear that \(h = \kappa\) or \(h = \kappa - 1\) according as \(n\) is even or odd. We consider these two cases separately.

(i) \(n\) even. Put

\[(2.4) \quad \begin{align*}
    R_1 &= (d_f - 1 \mid 1 \leq f \leq h), & \delta_h > 1, \\
    R_2 &= (d_f - 1 \mid 1 \leq f \leq h - 1), & \delta_h = 1,
\end{align*}\]

and \(R_\alpha = (e_g \mid 1 \leq g \leq h)\). Since

\[
\sum_{f=1}^{h} (d_f - 1) + \sum_{g=1}^{h} e_g = \sum_{f=1}^{h} (d_f - 1)/2 + \sum_{g=1}^{h} (e_g + 1)/2 = n/2,
\]

it follows that \([R_1, R_\alpha]\) is a diagonal partition of \(n/2\).

This correspondence is clearly reversible, so the total number of partitions \(D_\alpha\) with odd summands is equal to the number of partitions \(D_{n/2}\), which is \(p(n/2)\). In reversing the correspondence there is one small point to be noted. We define \([R_1, R_2]\) as before, where \(R_2\) has \(h\) summands. If \(R_1\) has \(h - 1\) summands, we then put \(Q_1 = (d_1, \ldots, d_{h-1}, 1)\), i.e.,

\[(2.5) \quad d_h = \delta_h = 1.\]

(ii) \(n\) odd. Define \(R_1\) as in case (i) and let

\[(2.6) \quad R_2 = (e_g \mid 2 \leq g \leq \kappa).\]
Note that if $k=1$, both $R_1$ and $R_2$ are the empty partition. We have, for $k \geq 2$,
\[
\sum_{i=1}^{k} (d_i - 1) + \sum_{g=2}^{k} e_g = \sum_{i=1}^{k} (d_i - 1)/2 + \sum_{g=2}^{k} (e_g + 1)/2.
\]
\[
= (n - e_1)/2 + (k - 1 - h)/2 = (n - e_1)/2 = (n - 1)/2 - (e_1 - 1).
\]
Hence $[R_1, R_2]$ corresponds by (2.1) to a partition
\[(2.7) \quad P' = (x_1, \ldots, x_m)\]
of $(n-1)/2 - (e_1 - 1)$. By (2.2) and (2.6), the largest summand of $P'$ is $x_1 = e_2$ (if $k \geq 2$; otherwise $P'$ is empty). Since $e_1 - 1 \leq e_2$ by (2.3),
\[(2.8) \quad P = (e_1 - 1, x_1, \ldots, x_m)\]
is a partition of $(n-1)/2$ normalized by (1.3). Associating $P$ with $[Q_1, Q_2]$, we obtain a correspondence between the diagonal partitions of $n$ with odd summands and the partitions of $(n-1)/2$.

Actually, the results proved in this section are contained in a more general theorem of Sylvester [3, pp. 283-285], but it seems appropriate to reprove the special cases needed here, since Sylvester’s paper is relatively unknown and suffers from an unduly complicated notation. It is to be noted, furthermore, that these correspondences give the results obtained in [1] by the use of generating functions where, however, the identity attributed to Durfee (p. 871) is really due to Euler [2, Theorem 351]. Also the numbers denoted there by $d$, $e$, $h$, evidently correspond to those denoted in this section by the same symbols.

We now write $\Delta$ for a $\delta$-partition, i.e., a solution of (1.1) and (1.2). If $S$ is the set (with multiplicities) of the summands of $D$ or $\Delta$, we write $D = D^S$, $\Delta = \Delta^S$. The results of this section show that Theorems 1 and 2 are an immediate consequence of the following.

**Lemma.** There is a 1-1 correspondence between the partitions $D^S$ and $\Delta^S$ for any fixed set $S$ of summands.

This is a strengthened form of the lemma (not explicitly labeled) in [1, pp. 870-871], where it is proved that the two classes of partitions in question are equinumerous. For his sets $A'$ and $B'$ (p. 871) evidently correspond to our $Q_1$ and $Q_2$.

3. **Proof of the lemma.** (i) Given $D^S = [Q_1, Q_2]$, we determine the corresponding $\Delta^S$ as follows. Put
\[(3.1) \quad \alpha_{2^j} = d_j, \quad \alpha_{2^j-1} = e_g, \quad h + \kappa = \nu,\]
where \( d_t, e_q \) are given by (2.1). Thus the \( D^s \) correspond to the sequences \( \{a_t | 1 \leq t \leq v\} \), where the notation is self-explanatory. We then set

\[
\alpha_t = a_t \iff \alpha_t > \alpha_{t+1}; \quad t = t_i \quad (1 \leq i \leq r - 1),
\]

\[
\alpha_t = b_j \iff \alpha_t \leq \alpha_{t+1}; \quad t = u_j \leq v - 1 \quad (1 \leq j \leq s),
\]

\[
\alpha_t = a_r, \quad r + s = v = t_r,
\]

in which \( \{t_i | 1 \leq i \leq r\} \) and \( \{u_j | 1 \leq j \leq s\} \) are increasing sequences. This gives

\[
\Delta^s = \sum_i a_i + \sum_j b_j.
\]

We prove now that \( \Delta^s \) satisfies (1.1) and (1.2). Recalling that \( Q_1 \) and \( Q_2 \) are partitions into distinct summands by (2.1), we see by (3.1) that \( \alpha_t > \alpha_{t+1} \). Hence \( b_j \) and \( b_{j+1} \) are never consecutive terms of \( \{\alpha_t\} \). Suppose \( \alpha_t = a_i, i \leq r - 1 \), so that \( a_i > a_{i+1} \). Then \( a_{i+1} = \alpha_{i+1} \) or \( \alpha_{i+1} \), and \( a_i > a_{i+1} \). If \( \alpha_t = b_j \), then \( b_{j+1} = \alpha_{j+1} \) or \( b_{j+1} = \alpha_{j+1} \). Hence \( b_j \leq a_i \). Thus \( b_j > b_{j+1} \).

By the remark above, \( u_j \geq u_{j-1} + 2 \). Since \( u_1 \geq 1 \), this implies that \( u_j \geq 2j - 1 \). Then by (3.2), \( 2j - 1 \leq v - 1, j \leq s \leq v/2 \) and \( r + s = v \), so that \( s \leq r \). If \( \alpha_t = b_j \), then \( \alpha_{j+1} = a_i \) where \( i = u_j + 1 - j \). Hence \( b_j \leq a_i \). The proof is complete.

(ii) Given \( \Delta^s \), we obtain the corresponding \( D^s \). Put \( r + s = v \) and define

\[
i_* = \text{Max} (b_i \leq a_i),
\]

\[
i_j = \text{Max} (i < i_{j+1} \exists b_j \leq a_i, \quad 1 \leq j \leq s - 1).
\]

Since \( b_j \leq a_i \) by hypothesis, the numbers \( i_j \) exist. Now form the sequence \( \{\beta_v | 1 \leq v \leq v \} \), in which

\[
\beta_v = a_{i^*}; \quad \beta_{v+1} = a_i, \quad i = i_j \rightarrow \beta_v = b_j \rightarrow \beta_{v-1} = a_{i-1},
\]

\[
\beta_{v+1} = a_i, \quad i \neq i_j \rightarrow \beta_v = a_{i-1},
\]

where \( v \leq v - 1 \) and \( a_{i-1} \) is absent for \( i = 1 \).

Now any sequence \( \{a_t\} \) which consists of the elements of \( S \) in some order and for which \( \alpha_t > \alpha_{t+1} \) corresponds to a partition \( D^s \) by (3.1). We show that \( \beta_v > \beta_{v+1} \leq \) so that the sequences \( \{\beta_v\} \) are included in the set of \( \{\alpha_t\} \). Hence (3.5) and (3.1) give \( D^s \).

Since the numbers \( i_j \) are distinct, \( b_j \) and \( b_{j+1} \) are not consecutive terms of \( \{\beta_v\} \).Moreover \( \{a_i\} \) and \( \{b_i\} \) are strictly decreasing by (1.2). Suppose that \( \beta_v = b_j \), then \( \beta_{v+1} = a_i \) and \( \beta_{v-1} = a_{i-1} \) for \( i = i_j \). Now
if $\beta_{++} = a_{++}$, $b_j > a_{++}$ by the maximal property of $i_i$. Suppose that $\beta_+ = a_{-+}$, $\beta_+ = b_i$ so that $\beta_+ = a_{-+}$. Then $b_j \leq a_i < a_{-+}$. Hence $\beta_+ > \beta_{++}$ in both cases. In the remaining case, $\beta_+$ and $\beta_{++}$ are both $a$'s or both $b$'s and the result is evident.

Now let $U$ be the mapping defined by (3.4) and (3.5), and $T$ that of (3.2), where $T$ and $U$ are evidently one-to-one. Since we have proved that $\Delta^\delta$ given by (3.3) satisfies the definition in §1, we can apply $U$ to (3.3). It is easily seen that $U$ is the inverse of $T$, so that the sequences $\{\alpha_i\}$ are included in the set of $\{\beta_i\}$. Hence the two sets are identical, for the converse has already been proved. We have therefore obtained the required 1-1 correspondence between the diagonal partitions $D^\delta$ and the $\delta$-partitions $\Delta^\delta$, which proves the lemma.

For completeness, we indicate how a proof of the lemma can be obtained by applying more directly the method of [1]. As explained there, it suffices to consider the case where the elements of the summand set $S$ are distinct. Hence we can assume that $S$ is the set of the first $k$ integers, so that $r + s = k$ in (1.1). Denote by $Q(k, m)$ (p. 870) the number of $\Delta^\delta$ for which $r - s \geq m$. Thus $k - 2s \geq m$, and $0 \leq s \leq \sigma = [(k - m)/2]$. Now we put $Q(k, m) = N_k(\sigma)$, the number of $\delta$-partitions formed from the first $k$ integers for which $s \leq \sigma$. This definition leads to a simpler and more natural recurrence relation for this function than that given in the reference.

We have $b_s = 1$ or $a_s = 1$, where $b_s = 1$ if $r = s$. If $\sigma < k/2$, we have $N_{k-1}(\sigma)$ $\delta$-partitions with $a_s = 1$, since $r > s$ in this case, and $N_{k-1}(\sigma - 1)$ of them with $b_s = 1$, since removal of the 1 gives a $\delta$-partition in both cases. Thus for $\sigma < k/2$ we have

\[(3.6) \quad N_k(\sigma) = N_{k-1}(\sigma) + N_{k-1}(\sigma - 1); \quad k \geq 2,\]

in which $\sigma \leq (k - 1)/2$ so that each term of (3.6) is defined. If $\sigma > k/2$, define $N_k(\sigma) = N_k(k - \sigma)$. Then (3.6) holds. The remaining case is when $\sigma = k/2$, in which the number of $\delta$-partitions in question with $a_s = 1$ is $N_{k-1}(\sigma - 1)$, since $r > s$. Here $\sigma - 1 = (k - 1) - \sigma$, so that $N_{k-1}(\sigma - 1) = N_{k-1}(\sigma)$ by our definition. Now the number of $\Delta^\delta$ with $b_s = 1$ is still $N_{k-1}(\sigma - 1)$ and we conclude that (3.6) holds for all $\sigma$.

Since $N_k(0) = 1$ ($k \geq 1$) and $N_1(1) = 1$, (3.6) gives $N_k(\sigma) = \frac{1}{k}C_\sigma$. Now $\kappa C_\sigma$ is also the number of subsets of $S$ with cardinality $\sigma$, i.e., the number of separations of $S$ into two disjoint subsets $A$ and $B$, where $A$ has cardinality $\sigma$. Hence we seek a 1-1 correspondence between the $\delta$-partitions $\Delta^\delta$ with at most $\sigma$ summands in the second row and the ordered pairs $[A, B]$. This can be done using (3.6) and leads to a correspondence similar to that of this section.
4. Examples. The correspondence giving the direct proof of Theorems 1 and 2 is illustrated with two numerical examples. Since we have already illustrated in §2 the graphical dissection of a $P_n$ into a $D_n$, the example illustrating Theorem 1 shows merely the correspondence between the partitions $D_n$ and $\Delta_n$ given by the construction used in the lemma. For Theorem 2 it suffices, as we have seen, to consider the correspondence between the $D_n$ and the $P_{[n/2]}$. Here we exemplify the most complicated case where $n$ is odd and $R_1$ [see (2.4)] has $h-1$ summands, the details of the other cases being similar but simpler.

We use the same notation as above except that we denote a diagonal partition with odd summands by $D^\odot$. All partitions or sequences occurring in a given example are in correspondence with each other, without explicit mention. For convenience, the partitions $D$ and $\Delta$ are written on two lines with the summands $d_i$ or $b_i$ on the lower line and with parentheses or commas omitted. In the second example, reference numbers refer to the relevant definitions in §2.

(i) Given $D$, to obtain $\Delta$ (and conversely).

a. $D = \begin{array}{cccc} 8 & 7 & 5 & 2 \\ 6 & 5 & 4 & 3 \end{array}$

b. $\{a_i\} = \{8 \ 6 \ 7 \ 5 \ 5 \ 4 \ 2 \ 3 \ 1\}$, where the $b_i$ of (3.2) are boldfaced.

c. $\Delta = \begin{array}{cccc} 8 & 7 & 5 & 4 & 3 \\ 6 & 5 & 2 \end{array}$

To reverse this procedure we write $\Delta$ in the form

$\begin{array}{cccc} 8 & 7 & 5 & 4 & 3 \\ 6 & 5 & 2 \end{array}$

in which each $b_i$ is aligned vertically with the corresponding $a_i$, $i = i_j$, of (3.4), and proceed as described there.

(ii) Given $P = P_{[n/2]}$, to obtain $D^\odot$, $n$ odd.

a. $P = (7 \ 6 \ 5 \ 5 \ 5 \ 1 \ 1) = (e_1 - 1, x_1, \ldots, x_6)$ (2.8),

b. $P' = (6 \ 5 \ 5 \ 5 \ 1 \ 1)$ (2.7),

c. $R_2 = 6 \ 4 \ 3 \ 2$, $h = 4, \kappa = 5$ (2.4), (2.6),

d. $R_1 = 5 \ 2 \ 1$, $D^\odot = 15 \ 11 \ 7 \ 5 \ 3 \ 1$ (2.3).
Here we have used (2.5).

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References


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MULTIPLE TRANSITIVITY OF PRIMITIVE PERMUTATION GROUPS

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Introduction. We wish to consider two theorems on permutation groups which are a generalization of a two-part theorem found in [1, pp. 66–67], a theorem concerned with a permutation group on a finite set. We shall remove the restriction of finiteness.

The following notation and definitions will be used in the discussion:

If \( Y \) is a set, then \( |Y| \) denotes the cardinal number of \( Y \). If \( g \) is a mapping of \( X \) into \( Z \) and \( x \in X \), then \( xg \) denotes the image of \( x \) under \( g \). If \( Y \subseteq X \), then \( (Y)g \) denotes the image of \( Y \) under \( g \). If \( G \) is a group and \( H \subseteq G \) and \( g \in G \), then \( H^g = g^{-1}Hg \). A permutation group \( G \) on a set \( X \) is said to be \( r \)-ply (or \( r \)-fold) transitive on \( Y \subseteq X \) (where \( r \) is a nonzero cardinal number and \( |Y| \geq r \)), if (i) for every pair of subsets \( A \) and \( B \) of \( X \) of cardinal number \( r \) and for any one-to-one map \( f \) of \( A \) onto \( B \), there exists \( g \in G \) such that \( g|_A = f \) (where \( g|_A \) is the restriction of \( g \) to \( A \)), and if (ii) \( (Y)g = Y \) for every \( g \in G \). A 1-fold transitive group on \( Y \subseteq X \) is called transitive on \( Y \).

A permutation group \( G \neq 1 \) on a set \( X \) is said to be imprimitive on \( Y \subseteq X \) if: (i) \( Y \) can be written as a disjoint union of two or more nonvoid sets \( \{S_\alpha \}_{\alpha \in \mathbb{A}} \), (ii) for at least one \( \alpha \in \mathbb{A}, |S_\alpha| \geq 2 \) and (iii) given \( g \in G \) and \( \alpha \in \mathbb{A} \), then \( (S_\alpha)g = S_\beta \) for some \( \beta \in \mathbb{A} \). The sets \( S_\alpha, \alpha \in \mathbb{A} \), are called imprimitivity sets of \( G \) on \( Y \). If \( G \) is not imprimitive on \( Y \), then \( G \) is called primitive on \( Y \).

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