SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A SIMPLE PROOF OF JACOBI'S TRIPLE PRODUCT IDENTITY

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Bellman remarks in [1, p. 42] that there are no simple proofs known of the complete triple product identity

\[ \prod_{n=0}^{\infty} (1 - x^{2n+1}) (1 + x^{2n+1}) (1 + x^{2n+1} z^{-1}) = \sum_{n=-\infty}^{\infty} x^n z^n \]

with \( z \neq 0 \) and \( |x| < 1 \).

However the two identities of Euler,

\[ \prod_{n=0}^{\infty} (1 + x^n) = \sum_{n=0}^{\infty} \frac{x^n (n-1/2)^{2n}}{(1 - x) \cdots (1 - x^n)}, \quad |x| < 1, \]

and

\[ \prod_{n=0}^{\infty} (1 + x^n)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(1 - x) \cdots (1 - x^n)}, \quad |x| < 1, |z| < 1, \]

are rather easily established [1, p. 49]. It does not seem to have been noticed that Jacobi's triple product identity follows simply from Euler's identities.

\[ \prod_{n=0}^{\infty} (1 + x^{2n+1}) = \sum_{n=0}^{\infty} \frac{x^n z^n}{(1 - x^2) \cdots (1 - x^{2n})} \quad \text{(by (E1))} \]

\[ = \sum_{n=0}^{\infty} \frac{x^n z^n}{\prod_{j=0}^{\infty} (1 - x^{2j+2})} \]

\[ = \prod_{j=0}^{\infty} (1 - x^{2j+2}) \sum_{n=-\infty}^{\infty} x^n z^n \prod_{j=0}^{\infty} (1 - x^{2n+2j+2}) \]

(all terms of the sum with negative \( n \) are zero)

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\prod_{j=0}^{\infty} \frac{1}{1 - x^{2j+1} z} = \frac{1}{\prod_{j=0}^{\infty} (1 - x^{2j+1} z^{-1})^{-1} \sum_{n=-\infty}^{\infty} x^{n} g^{n}} \quad \text{(using (E2) and replacing } \pi + m \text{ by } n \text{ in the inner sum).}
\]

The above argument is valid provided \( |x| < |z| \). The complete result for all nonzero \( z \) follows by analytic continuation.

**Bibliography**