UNIFORM INTEGRABILITY AND THE POINTWISE ERGODIC THEOREM

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Let $(X, \mathcal{B}, m)$ be a finite measure space. We shall denote by $L^p(m)$ $(1 \leq p < \infty)$ the Banach space of all real-valued $\mathcal{B}$-measurable functions f defined on X such that $|f|^p$ is m-integrable, and by $L^\infty(m)$ the Banach space of all real-valued, $\mathcal{B}$-measurable, m-essentially bounded functions defined on X; as usual, the norm in $L^p(m)$ is given by $\|f\|_p = \{\int_X |f(x)|^p \, dm\}^{1/p}$, and the norm in $L^\infty(m)$ by $\|g\|_\infty = m\text{-ess. sup}_{x \in X} |g(x)|$. Two functions in $L^p(m)$ or $L^\infty(m)$ will be identified if they differ only on a set of m-measure zero. In this note, we shall be concerned with a *positive* linear operator T of $L^1(m)$ into $L^1(m)$ with $\|T\|_1 \leq 1$.

We say that *the pointwise ergodic theorem* (*the $L^1(m)$-mean ergodic theorem*, respectively) *holds* for such an operator T if for every f in $L^1(m)$, the sequence of averages $\{1(/n) \sum_{k=0}^{n-1} T^k f\}$ converges m-almost everywhere (in the norm of $L^1(m)$, respectively) to a function in $L^1(m)$. Recently, R. V. Chacon [1] constructed a class of positive linear operators in $L^1(m)$ with the norm equal to 1 for which the pointwise ergodic theorem fails to hold. Also, A. Ionescu Tulcea [5], [6] showed that in the group of all positive invertible linear isometries of $L^1(m)$ the set of all T for which the pointwise ergodic theorem fails to hold forms a set of second category with respect to the strong operator topology. On the other hand, the ergodic theorem of Hopf-Dunford-Schwartz [4] tells us that if, in addition, T maps $L^\infty(m)$ into $L^\infty(m)$ and $\|T\|_\infty \leq 1$, then the pointwise ergodic theorem is valid for such T. In view of these facts, it is interesting to find out what other additional conditions on T would guarantee the validity of the pointwise ergodic theorem. In this note, we shall find a few such conditions which are weaker than the condition of the Hopf-Dunford-Schwartz theorem (though our conditions seem to work for a finite measure space only). We also obtain a result (corollary to Theorem 1, below) which generalizes a result obtained by N. Dunford and D. S. Miller in [3].

First of all, let us observe that if our operator T satisfies $\|T\|_1 < 1$, then the pointwise ergodic theorem is always valid. This is because, for such an operator T, $\sum_{n=0}^{\infty} |T^n f(x)| < \infty$ m-almost everywhere for every f in $L^1(m)$, since

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$$\int_X \sum_{n=0}^{\infty} | T^n f(x) | \, dm = \sum_{n=0}^{\infty} \|T^n f\|_1 \leqq \|f\|_1 \cdot \sum_{n=0}^{\infty} \|T\|_1^n = \frac{\|f\|_1}{1 - \|T\|_1} < \infty.$$

Therefore, we shall assume from now on that T satisfies $\|T\|_1 = 1$.

Let us denote by U the adjoint operator of T. Then, U is a linear operator mapping $L^\infty(m)$ into $L^\infty(m)$, and for every pair of functions f, g with f in $L^1(m)$, g in $L^\infty(m)$,

$$\int_X g(x) Tf(x) \, dm = \int_X f(x) Ug(x) \, dm$$

holds. It is clear that U is also a positive operator and that $\|U\|_\infty = \|T\|_1 = 1$.

We say that a sequence of functions $\{f_n\}$ in $L^1(m)$ is *uniformly integrable* if

$$\lim_{N \to \infty} \int_{\{|f_n(x)| > N\}} | f_n(x) | \, dm = 0 \quad \text{uniformly in } n.$$

A subset K of a Banach space E is called *weakly sequentially compact* if every sequence $\{\phi_n\}$ of elements in K contains a subsequence which converges weakly to an element in E. It is well known that a sequence $\{f_n\}$ in $L^1(m)$ is uniformly integrable if and only if the set $\{f_n\}$ is weakly sequentially compact in $L^1(m)$.

We shall need the following

LEMMA 1. *Let $\{f_n\}$ and $\{g_n\}$ be two sequences of functions in $L^1(m)$ such that, for each n, $|g_n(x)| \leqq |f_n(x)|$ holds m-almost everywhere. Suppose $\{f_n\}$ is uniformly integrable, then so is $\{g_n\}$.*

PROOF. Obvious from the following inequalities

$$\int_{\{|g_n(x)| > N\}} | g_n(x) | \, dm \leqq \int_{\{|g_n(x)| > N\}} | f_n(x) | \, dm \leqq \int_{\{|f_n(x)| > N\}} | f_n(x) | \, dm.$$

The main result of this note is the following

THEOREM 1. *Suppose for our operator T, the sequence of functions $\{(1/n) \sum_{k=0}^{n-1} T^k 1\}$, where $1(x)$ denotes the function taking constant value one m-almost everywhere, is uniformly integrable; then, the pointwise ergodic theorem holds for T.*

For the proof of this theorem, we shall need a few more lemmas.

LEMMA 2. *Suppose the hypothesis of Theorem 1 is satisfied for T, then the $L^1(m)$-mean ergodic theorem holds for T.*

Proof. Let, for any set B in $\mathcal{B}$, $\chi_B(x)$ denote its characteristic function. By the positivity of T, we have $0 \leqq T^k\chi_B(x) \leqq T^k 1(x)$ m-almost everywhere for each k. Therefore, for each n,

$$0 \leqq \frac{1}{n}\sum_{k=0}^{n-1} T^k\chi_B(x) \leqq \frac{1}{n}\sum_{k=0}^{n-1} T^k 1(x)$$

holds m-almost everywhere. Since the set of all characteristic functions of sets in $\mathcal{B}$ forms a fundamental subset of $L^1(m)$, Lemma 1 and our hypothesis imply that the set $\{(1/n)\sum_{k=0}^{n-1} T^k f\}$ is weakly sequentially compact in $L^1(m)$ for every f belonging to a fundamental subset of $L^1(m)$. Since it is clear that $\|(1/n)\sum_{k=0}^{n-1} T^k\|_1 \leqq 1$ for all n, the mean ergodic theorem of Yosida and Kakutani [7] now implies that the $L^1(m)$-mean ergodic theorem must hold for our T. Q.E.D.

Suppose now the $L^1(m)$-mean ergodic theorem is valid for T. Let us write

$$h = s\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k 1,$$

where s-lim means the limit taken in the sense of the norm in $L^1(m)$. Then, it is clear that

(i) $h \in L^1(m)$,

(ii) $h(x) \geqq 0$ m-almost everywhere,

(iii) $Th(x) = h(x)$ holds m-almost everywhere (i.e., h is invariant under T).

Furthermore, since the norm-convergence implies the weak convergence, we have

(iv) for every set B in $\mathcal{B}$,

$$\int_B h(x)\,dm = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\int_B T^k 1(x)\,dm = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_X U^k\chi_B(x)\,dm.$$

Lemma 3. *Let us denote $A = \{x \mid h(x) = 0\}$, where $h(x)$ is the function defined above. Then, $U\chi_A(x) \leqq \chi_A(x)$ holds m-almost everywhere.*

Proof. Since U satisfies $\|U\|_\infty = 1$, it follows that $U1(x) \leqq 1$ m-almost everywhere. Now, the positivity of U implies $U\chi_A(x) \leqq U1(x) \leqq 1$ holds m-almost all x. Therefore, for m-almost all x in A, $U\chi_A(x) \leqq \chi_A(x)$ is satisfied. Now, suppose $x \in X - A$. We must prove that $U\chi_A(x) = 0$ holds for m-almost all such x. Suppose this were not the case. Then, there would exist a set $B \subset X - A$ such that $m(B) > 0$ and such that $U\chi_A(x) > 0$ holds for all x in B. But then, since $h(x) > 0$ for all $x \in B$ and $h(x)\,U\chi_A(x) \geqq 0$ for m-almost all $x \in X$, we get

$$0 < \int_B h(x) U\chi_A(x)\, dm \leqq \int_X h(x) U\chi_A(x)\, dm = \int_A Th(x)\, dm$$

$$= \int_A h(x)\, dm = 0,$$

a contradiction. Q.E.D.

Since U is order preserving, Lemma 3 implies that

$$\chi_A(x) \geqq U\chi_A(x) \geqq \cdots \geqq U^k\chi_A(x) \geqq \cdots$$

holds m-almost everywhere for the set A.

LEMMA 4. *Let A be the same set as in Lemma 3. Then,*

$$A \subset \bigcup_{k=1}^{\infty} \{x \mid U^k\chi_A(x) < \chi_A(x)\}$$

with the possible exception of a set of m-measure zero.

PROOF. Suppose false. Then, there would exist a set $C \subset A$ such that $m(C) > 0$ and such that $U^k\chi_A(x) \geqq \chi_A(x)$ holds for all x in C and for all k. Since $U^k\chi_A \leqq \chi_A(x)$ must hold m-almost everywhere for each k, this means we have $U^k\chi_A(x) = \chi_A(x) = 1$ for all k and for m-almost all x in C. But then, for each n,

$$0 < m(C) = \frac{1}{n}\sum_{k=0}^{n-1}\int_C dm = \frac{1}{n}\sum_{k=0}^{n-1}\int_C U^k\chi_A(x)\, dm$$

$$\leqq \frac{1}{n}\sum_{k=0}^{n-1}\int_X U^k\chi_A(x)\, dm = \frac{1}{n}\sum_{k=0}^{n-1}\int_A T^k 1(x) dm,$$

and therefore,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\int_A T^k 1(x)\, dm \geqq m(C) > 0.$$

But, since the left side of the last inequality equals $\int_A h(x) dm = 0$, we have a contradiction.

LEMMA 5. *For every f in $L^1(m)$, $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} T^k f(x) = 0$ m-almost everywhere on the set A.*

PROOF. Actually, we can prove even more: on the set A, $\sum_{n=0}^{\infty} T^n|f|(x) < \infty$ holds m-almost everywhere for every f in $L^1(m)$. In view of Lemma 4, it suffices to prove this fact on the set $\{x \mid U^k\chi_A(x) < \chi_A(x)\}$ for each fixed integer k. So, let us fix k. Then, for any $n > k$, we have

$$
\begin{aligned}
0 &\leqq \int_X \left(\sum_{j=0}^{n-1} T^j |f| (x) \right) (\chi_A(x) - U^k \chi_A(x))\, dm \\
&= \int_X |f| (x) \left\{ \sum_{j=0}^{n-1} U^j(\chi_A - U^k \chi_A)(x) \right\} dm \\
&= \int_X |f| (x) \{ \chi_A(x) + U\chi_A(x) + \cdots + U^{k-1}\chi_A(x) - U^n \chi_A(x) - \cdots \\
&\qquad\qquad - U^{n+k-1}\chi_A(x) \} \, dm \\
&\leqq 2k \int_X |f| (x)\, dm < \infty .
\end{aligned}
$$

Since this is true for any $n>k$, we must have $\sum_{j=0}^{\infty} T^j|f|(x) < \infty$ m-almost everywhere on the set

$$\{x \mid U^k\chi_A(x) < \chi_A(x)\}. \qquad \text{Q.E.D.}$$

Proof of Theorem 1. By Lemma 2, we know that there exists a non-negative function $h(x)$ which is invariant under T. For any f in $L^1(m)$, we now apply the general ergodic theorem of Chacon and Ornstein [2] to f and h. Then,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} T^k f}{\sum_{k=0}^{n-1} T^k h} = \frac{1}{h} \lim_{n\to\infty} \frac{1}{h} \sum_{k=0}^{n-1} T^k f$$

exists m-almost everywhere on the set $\{x \mid h(x) > 0\}$. Therefore, $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} T^k f(x)$ exists m-almost everywhere on the set $\{x \mid h(x) > 0\}$ for every f in $L^1(m)$. On the other hand, by Lemma 5, $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} T^k f(x) = 0$ m-almost everywhere on $\{x \mid h(x) = 0\}$ for every f in $L^1(m)$. Thus, the pointwise ergodic theorem is valid for T. Q.E.D.

Remark. If in Theorem 1, f in $L^1(m)$ is strictly positive m-almost everywhere, then $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} T^k f(x)$ is strictly positive m-almost everywhere on the set $\{x \mid h(x) > 0\}$, because if not, the limit of the ratio

$$\frac{\sum_{k=0}^{n-1} T^k h(x)}{\sum_{k=0}^{n-1} T^k f(x)}$$

would become unbounded on some part of the set $\{x \mid h(x) > 0\}$, contradicting the general ergodic theorem of Chacon-Ornstein.

The argument used in the proof above includes the following

COROLLARY. *If the $L^1(m)$-mean ergodic theorem holds for T, then the pointwise ergodic theorem is also true for T.*

We can also prove

THEOREM 2. *Suppose T satisfies the following additional condition: for some p, $1 < p < \infty$, T maps $L^p(m)$ into $L^p(m)$ and $\|T\|_p \leqq 1$. Then, the pointwise ergodic theorem is valid for T in $L^1(m)$.*

PROOF. Since $L^p(m)$, $1 < p < \infty$, is reflexive, any bounded subset is weakly sequentially compact. Therefore, the set $\{(1/n)\sum_{k=0}^{n-1} T^k 1\}$ is weakly sequentially compact in $L^p(m)$. Since $(X, \mathcal{B}, m)$ is a finite measure space, this implies that $\{(1/n)\sum_{k=0}^{n-1} T^k 1\}$ is weakly sequentially compact in $L^1(m)$ as well. Therefore, the proof of Theorem 2 is reduced to that of Theorem 1.

REMARK. Suppose as in the Hopf-Dunford-Schwartz theorem, T maps $L^\infty(m)$ into $L^\infty(m)$ and $\|T\|_\infty \leqq 1$. Then, for each n,

$$0 \leqq \frac{1}{n} \sum_{k=0}^{n-1} T^k 1(x) \leqq 1 \qquad m\text{-almost everywhere}$$

so that it is clear $\{(1/n)\sum_{k=0}^{n-1} T^k 1\}$ is uniformly integrable in this case since $m(X) < \infty$. Thus, the hypothesis of our Theorem 1 is weaker than that of Hopf-Dunford-Schwartz. Also, by the Riesz convexity theorem, $\|T\|_1 \leqq 1$, $\|T\|_\infty \leqq 1$ would imply that T maps $L^p(m)$ into $L^p(m)$ for each p, $1 < p < \infty$, and $\|T\|_p \leqq 1$. Therefore, the hypothesis of our Theorem 2 is also weaker.

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