A REMARK CONCERNING LITTLEWOOD'S
TAUBERIAN THEOREM

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Let \{a_n\}, n = 0, 1, \ldots, be real numbers such that the series \(\sum_0^\infty a_n\) is Abel summable to \(s\). Then, by a well-known theorem of Littlewood, if

\[
a_n = O\left(\frac{1}{n}\right),
\]

the series converges to \(s\). Professor Carleson has suggested to the author the question of whether the condition

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

(which plainly implies that \(\sum a_n\) is Abel summable), implies the convergence of \(\sum a_n\) under a substantially weaker Tauberian condition than (1). (Such a theorem would have useful applications in the theory of Fourier series.) We show, however, the following negative result:

**Theorem.** Given any \(\varepsilon > 0\), there is a sequence \(a_n\) satisfying (2) with

\[
a_n = O\left(\frac{1}{n^{1-\varepsilon}}\right)
\]

for which \(\sum a_n\) is divergent.

**Proof.** We take \(a_0 = 0, a_n = n^{-\alpha} \cos \beta n\) for \(n = 1, 2, \ldots\), where \(\beta = 1/2k\) (\(k\) a positive integer) and \(\alpha = 1 - \beta\). Note first that the series \(\sum a_n\) diverges because the sum of consecutive terms in a block of terms having like sign does not tend to zero, as is readily verified. Hence the theorem will be proved if for every \(k\) the function

\[
F(y) = \sum_{n=1}^{\infty} n^{-\alpha} \cos \beta n^y
\]

is of bounded variation (B. V.) on \([0, \infty)\). Setting

\[
G(t, y) = t^{-\alpha} \cos \beta e^{-ty},
\]

we have

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258
\[ F(y) = \sum_{n=1}^{\infty} G(n, y) = \int_{0}^{\infty} G(t, y) \, dt = -\int_{1}^{\infty} [t] \, \frac{\partial G(t, y)}{\partial t} \, dt \]

\[ = G(1, y) + \int_{1}^{\infty} G(t, y) \, dt + \int_{1}^{\infty} (t - [t]) \, \frac{\partial G(t, y)}{\partial t} \, dt \]

and we show that each of the three terms on the right is B. V.

(i) \( G(1, y) = e^{-y} \in B. V. \)

(ii) Consider next the last term on the right. Call it \( H(y) \). Then

\[ H'(y) = \int_{1}^{\infty} (t - [t]) \, \frac{\partial^2 G(t, y)}{\partial t \partial y} \, dt, \]

\[ \int_{0}^{\infty} |H'(y)| \, dy \leq \int_{0}^{\infty} \int_{1}^{\infty} \left| \frac{\partial^2 G(t, y)}{\partial t \partial y} \right| \, dt \, dy. \]

From the estimate \( |\partial^2 G(t, y)/\partial t \partial y| \leq (2t^{2s-1} + \gamma t^s) e^{-ty} \quad (t \geq 1) \), the finiteness of the double integral follows, and so \( H \in B. V. \)

(iii) We are left finally with \( \int_{0}^{\infty} G(t, y) \, dt \), and it suffices to study instead \( \int_{0}^{1} G(t, y) \, dt \), since

\[ \int_{0}^{\infty} \left| \frac{\partial}{\partial y} \int_{0}^{1} G(t, y) \, dt \right| \, dy \leq \int_{0}^{\infty} \int_{0}^{1} t^{-a} e^{-ty} \, dt \, dy < \infty. \]

Now, \( \int_{0}^{\infty} G(t, y) \, dt = \int_{0}^{1} t^{-a} \cos t^b e^{-ty} \, dt = 2k \int_{0}^{1} e^{-u^b} \cos zu \, du \), where we have set \( u = (ty)^a \), \( z = y^{-b} \), and \( b^{-1} = 2k \). It is enough to verify that the last integral represents a function of \( B. V. \) for \( 0 \leq z < \infty \), and for this it suffices to remark that its derivative is of class \( L^1 \), since, in fact, both \( \int_{0}^{1} e^{-u^b} \cos zu \, du \) and \( \int_{0}^{1} e^{-u^b} \sin zu \, du \) fall off at \( \infty \) faster than any power of \( z^{-1} \), being Fourier transforms of functions on \((-\infty, \infty)\) possessing \( L^1 \) derivatives of every order. The theorem is proved.

Remark. The divergence of the above series, as well as the fact that it is Abel summable, follows from [1, Theorem 84], which implies that for any \( k > -1 \), \( \sum_{1}^{\infty} n^{-a} e^{in\beta} \) is summable \((C, k)\) if and only if \((k+1)\beta + \alpha > 1\).

Bibliography