IDEALS OF SQUARE SUMMABLE POWER SERIES. II

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The closed invariant subspaces of multiplication by $z$ in $H^2$ were determined by Beurling [1, Theorem IV, p. 253]. Vector generalizations of this theorem are known (Halmos [3] and the author [6]), but they involve an unnecessary use of analysis. We can now prove the theorem of [6] by purely algebraic and geometric methods. To emphasize these methods, we work with sequences, which we write as formal power series, rather than functions analytic in the unit disk.

Let $C$ be a Hilbert space with elements denoted by $a$, $b$, $c$, $\cdots$, and with norm $|\cdot|$. If $b$ is a vector in $C$, then $b$ is the linear functional on $C$ such that $\langle ba, a \rangle$ for every $a$ in $C$. A formal power series is a sequence $(a_0, a_1, a_2, \cdots)$ written $f(z) = \sum a_n z^n$ with an indeterminate $z$. Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be formal power series with coefficients $a_n$ and $b_n$ in $C$; let $B(z) = \sum B_n z^n$ be a formal power series whose coefficients $B_n$ are (bounded) operators in $C$; let $\alpha$ be a complex number, and let $c$ be a vector in $C$. Then $f(z) + g(z)$, $af(z)$, $cf(z)$, and $B(z)f(z)$ are the formal power series $\sum (a_n + b_n) z^n$, $\sum (\alpha a_n) z^n$, $\sum (\alpha c_n) z^n$, and $\sum (\sum_{k=0}^{n} B_k a_{n-k}) z^n$, respectively. A sequence $(f_k(z))$ of formal power series with coefficients in $C$ is said to be formally convergent if, for each $n = 0, 1, 2, \cdots$, the corresponding sequence of $n$th coefficients is convergent. Let $C(z)$ be the Hilbert space of formal power series $f(z) = \sum a_n z^n$ with coefficients $a_n$ in $C$, such that

$$||f||^2 = \sum |a_n|^2 < \infty.$$  

An ideal of $C(z)$ is a subspace $J$ of $C(z)$ which contains $sf(z)$ whenever it contains $f(z)$. The problem is to determine the closed ideals of $C(z)$. Let $B(z)$ be a formal power series whose coefficients are operators on $C$, and let $N(B)$ be the set of vectors $c$ in $C$ such that $B(z)c = 0$ identically. We write $J(B)$ for the set of all products $B(z)f(z)$ with $f(z)$ in $C(z)$. The relevant condition on $B(z)$ is:

1. $(Bz)(c_n)$ is an orthonormal set in $C(z)$ whenever $(c_n)$ is a sequence of unit vectors in $C$ orthogonal to $N(B)$.

It is not difficult to show that (1) is equivalent to the assertion that $f(z) \rightarrow B(z)f(z)$ is a partial isometry of $C(z)$ into itself.

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Theorem. A subset $\mathfrak{M}$ of $\mathcal{C}(z)$ is a closed ideal if and only if $\mathfrak{M} = \mathfrak{M}(B)$ for some formal power series $B(z)$ with operator coefficients which satisfies (1).

Proof. If a series $\sum f_n(z)$ converges in the metric of $\mathcal{C}(z)$, then it converges formally, and the two limits coincide. For, the linear transformation from $\mathcal{C}(z)$ to $\mathfrak{C}$, which takes a formal power series into its $n$th coefficient, is continuous.

Let $B(z)$ satisfy (1) and let $g(z) = B(z)f(z)$ be any element of $\mathfrak{M}(B)$. If $f(z) = \sum a_nz^n$ is chosen so that its coefficients are orthogonal to $N(B)$, then $\sum a_nB(z)a_n$ converges in the metric of $\mathcal{C}(z)$, and its norm squared is

$$\sum \|z^nB(z)a_n\|^2 = \sum \|a_n\|^2 = \|f\|^2.$$

But $\sum z^nB(z)a_n$ converges formally to $g(z)$, so by the above remark, $g(z)$ is in $\mathcal{C}(z)$ and $\|g\| = \|f\|$. We conclude easily that $\mathfrak{M}(B)$ is a closed ideal of $\mathcal{C}(z)$.

Conversely, let $\mathfrak{M}$ be a closed ideal of $\mathcal{C}(z)$ and let $\mathfrak{B}$ be the orthogonal complement in $\mathfrak{M}$ of the series $zf(z)$ with $f(z)$ in $\mathfrak{M}$. We begin by showing that the dimension of $\mathfrak{B}$ is no more than that of $\mathcal{C}$. Since $\mathfrak{B}$ is a closed subspace of $\mathcal{C}(z)$, the assertion is clear when $\mathfrak{B}$ has infinite dimension. For in this case, $\mathfrak{B}$ and $G(z)$ have the same dimension. Now let $\mathcal{C}$ have finite dimension $r$ and let $c_1, \ldots, c_r$ be an orthonormal basis for $\mathcal{C}$. Suppose, to the contrary, that the dimension of $\mathfrak{B}$ is greater than the dimension of $\mathfrak{B}$. Then there is an orthonormal set $f_0(z), \ldots, f_r(z)$ in $\mathfrak{B}$ containing $r+1$ elements. By the definition of $\mathfrak{B}$,

$$\langle z^m f_i(z), z^n f_j(z) \rangle = \delta_{mn}\delta_{ij},$$

$m, n = 0, 1, 2, \ldots$, and $i, j = 0, 1, 2, \ldots, r$. Let $M$ be the $(r+1) \times (r+1)$ matrix whose $i$th row vector is

$$X_i = (\mathcal{C}_f(z), \mathcal{C}_f(z), \ldots, \mathcal{C}_f(z), \mathcal{C}_f(z)).$$

Since the first and last columns of $M$ coincide, $\det M = 0$. A standard argument [4, pp. 23-24] shows that there are cofactors $F_0(z), \ldots, F_r(z)$ of $M$, not all identically zero, such that

$$F_0(z) \cdot X_0 + \cdots + F_r(z) \cdot X_r = 0.$$

The cofactors of $M$ are formal power series with square summable complex coefficients. To prove this, it suffices to show that if $c_j$ is any $c_j$, and if $f(z)$ is any $f_j(z)$, then $F(z)c_j f(z)$ is in $\mathfrak{K}(z)$ (where $\mathfrak{K}$ de-
notes the complex numbers) whenever $F(z)$ is in $\mathcal{K}(z)$. If $F(z) = \sum \alpha_n z^n$, then

$$F(z)\overline{f(z)} = \sum \alpha_n z^n \overline{f(z)}$$

holds with convergence in the formal sense. By (2),

$$\left\| \sum_{m}^{N} \alpha_n z^n \overline{f(z)} \right\|_{\mathcal{K}(z)}^{2} \leq |c|^2 \left\| \sum_{m}^{N} \alpha_n z^n f(z) \right\|^2_{\mathcal{C}(z)} = \sum_{m}^{N} |\alpha_n|^4.$$

It follows that the series in (4) converges in the metric of $\mathcal{K}(z)$, and by the assertion at the beginning of the proof, (4) holds in the metric of $\mathcal{K}(z)$. In particular, $F(z)\overline{f(z)}$ is in $\mathcal{K}(z)$. Therefore, the cofactors of $M$ have square summable coefficients.

Let $F_i(z) = \sum \alpha_n z^n$. By (3),

$$\sum_{i=0}^{r} F_i(z) \cdot f_i(z) = 0$$

and

$$\sum [\alpha_0 z^0 f_0(z) + \cdots + \alpha_r z^r f_r(z)] = 0,$$

with convergence of this last series in the metric of $\mathcal{C}(z)$. By (2) we obtain the contradiction that the norm squared of the left-hand side of (5) is

$$\sum (|\alpha_0|^2 + \cdots + |\alpha_r|^2) > 0.$$

This completes the proof that the dimension of $\mathfrak{B}$ is at most the dimension of $\mathcal{C}$.

If $\mathfrak{M}$ is the zero ideal, then $\mathfrak{M}$ is of the form $\mathfrak{M}(B)$ trivially. If $\mathfrak{M}$ contains a nonzero element, so does $\mathfrak{B}$. Let $(f_i(z))_{i \in I}$ be an orthonormal basis for $\mathfrak{B}$. Since the dimension of $\mathfrak{B}$ is no more than the dimension of $\mathcal{C}$, there is an orthonormal set $(c_i)_{i \in I}$ in $\mathcal{C}$ of the same cardinality. Define a formal power series $B(z)$ with operator coefficients by

$$B(z) = \sum \bar{c}_i f_i(z).$$

It is easy to see that $B(z)$ satisfies (1) with $N(B)$ equal to the orthogonal complement of the $c_i$ ($i \in I$), and that $\mathfrak{M}(B) \subseteq \mathfrak{M}$.

To complete the proof, we will show that the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}(B)$ in $\mathfrak{M}$ is zero. Notice that $\mathfrak{B} \subseteq \mathfrak{M}(B)$ and

$$\mathfrak{N} = \mathfrak{M}(B) \perp \mathfrak{M} \subseteq \mathfrak{B} \perp \mathfrak{M}.$$

Let $f(z)$ be any element of $\mathfrak{N}$. Then $f(z)$ is in $\mathfrak{M}$ and it is orthogonal
to $\mathfrak{M}$. By the definition of $\mathfrak{M}$, $f(z) = zf_1(z)$ for some $f_1(z)$ in $\mathfrak{M}$. If $g(z)$ is any element of $\mathfrak{M}(B)$,
\[\langle f_1(z), g(z) \rangle = \langle zf_1(z), zg(z) \rangle = \langle f(z), g(z) \rangle = 0,\]
since $\mathfrak{M}(B)$ is an ideal. Therefore, $f_1(z)$ is in $\mathfrak{M}$. Continuing by induction, we see that for every $r = 1, 2, 3, \ldots$, there is an $f_r(z)$ in $\mathfrak{M}$ such that $f(z) = zf_r(z)$. It follows that $f(z) = 0$ identically. Therefore $\mathfrak{M} = (0)$ and $\mathfrak{M} = \mathfrak{M}(B)$ as asserted.

Added in proof. The methods of this paper have since been used to obtain the existence of invariant subspaces for transformations $T$ which are bounded by 1, when $1 - T^*T$ is completely continuous.

References


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