IDEALS OF SQUARE SUMMABLE POWER SERIES. II

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The closed invariant subspaces of multiplication by z in H^2 were determined by Beurling [1, Theorem IV, p. 253]. Vector generalizations of this theorem are known (Halmos [3] and the author [6]), but they involve an unnecessary use of analysis. We can now prove the theorem of [6] by purely algebraic and geometric methods. To emphasize these methods, we work with sequences, which we write as formal power series, rather than functions analytic in the unit disk.

Let $\mathcal{C}$ be a Hilbert space with elements denoted by $a, b, c, \cdots$, and with norm $|\cdot|$. If b is a vector in $\mathcal{C}$, then $\bar{b}$ is the linear functional on $\mathcal{C}$ such that $\bar{b}a = \langle a, b\rangle$ for every a in $\mathcal{C}$. A formal power series is a sequence $(a_0, a_1, a_2, \cdots)$ written $f(z) = \sum a_n z^n$ with an indeterminate z. Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be formal power series with coefficients a_n and b_n in $\mathcal{C}$; let $B(z) = \sum B_n z^n$ be a formal power series whose coefficients B_n are (bounded) operators in $\mathcal{C}$; let α be a complex number, and let c be a vector in $\mathcal{C}$. Then $f(z)+g(z)$, $\alpha f(z)$, $\bar{c}f(z)$, and $B(z)f(z)$ are the formal power series $\sum(a_n+b_n)z^n$, $\sum(\alpha a_n)z^n$, $\sum(\bar{c}a_n)z^n$, and $\sum(\sum_{k=0}^{n} B_k a_{n-k})z^n$, respectively. A sequence $(f_k(z))$ of formal power series with coefficients in $\mathcal{C}$ is said to be *formally convergent* if, for each $n=0, 1, 2, \cdots$, the corresponding sequence of nth coefficients is convergent. Let $\mathcal{C}(z)$ be the Hilbert space of formal power series $f(z) = \sum a_n z^n$ with coefficients a_n in $\mathcal{C}$, such that

$$\|f\|^2 = \sum |a_n|^2 < \infty.$$

An *ideal* of $\mathcal{C}(z)$ is a subspace $\mathfrak{M}$ of $\mathcal{C}(z)$ which contains $zf(z)$ whenever it contains $f(z)$. The problem is to determine the closed ideals of $\mathcal{C}(z)$. Let $B(z)$ be a formal power series whose coefficients are operators on $\mathcal{C}$, and let $N(B)$ be the set of vectors c in $\mathcal{C}$ such that $B(z)c=0$ identically. We write $\mathfrak{M}(B)$ for the set of all products $B(z)f(z)$ with $f(z)$ in $\mathcal{C}(z)$. The relevant condition on $B(z)$ is:

(1) *$(z^n B(z)c_n)$ is an orthonormal set in $\mathcal{C}(z)$ whenever (c_n) is a sequence of unit vectors in $\mathcal{C}$ orthogonal to $N(B)$.*

It is not difficult to show that (1) is equivalent to the assertion that $f(z)\to B(z)f(z)$ is a partial isometry of $\mathcal{C}(z)$ into itself.

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THEOREM. *A subset* $\mathfrak{M}$ *of* $\mathcal{C}(z)$ *is a closed ideal if and only if* $\mathfrak{M}=\mathfrak{M}(B)$ *for some formal power series* $B(z)$ *with operator coefficients which satisfies* (1).

PROOF. If a series $\sum f_n(z)$ converges in the metric of $\mathcal{C}(z)$, then it converges formally, and the two limits coincide. For, the linear transformation from $\mathcal{C}(z)$ to $\mathcal{C}$, which takes a formal power series into its nth coefficient, is continuous.

Let $B(z)$ satisfy (1) and let $g(z)=B(z)f(z)$ be any element of $\mathfrak{M}(B)$. If $f(z)=\sum a_n z^n$ is chosen so that its coefficients are orthogonal to $N(B)$, then $\sum z^n B(z)a_n$ converges in the metric of $\mathcal{C}(z)$, and its norm squared is

$$\sum \|z^n B(z) a_n\|^2 = \sum |a_n|^2 = \|f\|^2. \tag{1}$$

But $\sum z^n B(z)a_n$ converges formally to $g(z)$, so by the above remark, $g(z)$ is in $\mathcal{C}(z)$ and $\|g\|=\|f\|$. We conclude easily that $\mathfrak{M}(B)$ is a closed ideal of $\mathcal{C}(z)$.

Conversely, let $\mathfrak{M}$ be a closed ideal of $\mathcal{C}(z)$ and let $\mathcal{B}$ be the orthogonal complement in $\mathfrak{M}$ of the series $zf(z)$ with $f(z)$ in $\mathfrak{M}$. We begin by showing that the dimension of $\mathcal{B}$ is no more than that of $\mathcal{C}$. Since $\mathcal{B}$ is a closed subspace of $\mathcal{C}(z)$, the assertion is clear when $\mathcal{C}$ has infinite dimension. For in this case, $\mathcal{C}$ and $\mathcal{C}(z)$ have the same dimension. Now let $\mathcal{C}$ have finite dimension r and let $c_1, \cdots, c_r$ be an orthonormal basis for $\mathcal{C}$. Suppose, to the contrary, that the dimension of $\mathcal{B}$ is greater than the dimension of $\mathcal{C}$. Then there is an orthonormal set $f_0(z), \cdots, f_r(z)$ in $\mathcal{B}$ containing $r+1$ elements. By the definition of $\mathcal{B}$,

$$\langle z^m f_i(z),\, z^n f_j(z)\rangle = \delta_{mn}\delta_{ij}, \tag{2}$$

$m, n=0, 1, 2, \cdots$, and $i, j=0, \cdots, r$. Let M be the $(r+1)\times(r+1)$ matrix whose ith row vector is

$$X_i = (\bar{c}_1 f_i(z),\, \bar{c}_2 f_i(z),\, \cdots,\, \bar{c}_r f_i(z),\, \bar{c}_1 f_i(z)).$$

Since the first and last columns of M coincide, det $M=0$. A standard argument [4, pp. 23–24] shows that there are cofactors $F_0(z), \cdots, F_r(z)$ of M, not all identically zero, such that

$$F_0(z)\cdot X_0 + \cdots + F_r(z)\cdot X_r = 0. \tag{3}$$

The cofactors of M are formal power series with square summable complex coefficients. To prove this, it suffices to show that if c is any c_i and if $f(z)$ is any $f_j(z)$, then $F(z)\bar{c}f(z)$ is in $\mathcal{K}(z)$ (where $\mathcal{K}$ denotes the complex numbers) whenever $F(z)$ is in $\mathcal{K}(z)$. If $F(z) = \sum \alpha_n z^n$, then

$$F(z)\bar{c}f(z) = \sum \alpha_n z^n \bar{c}f(z) \tag{4}$$

holds with convergence in the formal sense. By (2),

$$\left\| \sum_M^N \alpha_n z^n \bar{c}f(z) \right\|^2_{\mathcal{K}(z)} \leqq |c|^2 \left\| \sum_M^N \alpha_n z^n f(z) \right\|^2_{\mathcal{C}(z)} = \sum_M^N |\alpha_n|^2.$$

It follows that the series in (4) converges in the metric of $\mathcal{K}(z)$, and by the assertion at the beginning of the proof, (4) holds in the metric of $\mathcal{K}(z)$. In particular, $F(z)\bar{c}f(z)$ is in $\mathcal{K}(z)$. Therefore, the cofactors of M have square summable coefficients.

Let $F_i(z) = \sum \alpha_{in} z^n$. By (3),

$$\sum_{i=0}^{r} F_i(z) \cdot f_i(z) = 0$$

and

$$\sum \left[\alpha_{0n} z^n f_0(z) + \cdots + \alpha_{rn} z^n f_r(z)\right] = 0, \tag{5}$$

with convergence of this last series in the metric of $\mathcal{C}(z)$. By (2) we obtain the contradiction that the norm squared of the left-hand side of (5) is

$$\sum (|\alpha_{0n}|^2 + \cdots + |\alpha_{rn}|^2) > 0.$$

This completes the proof that the dimension of $\mathcal{B}$ is at most the dimension of $\mathcal{C}$.

If $\mathfrak{M}$ is the zero ideal, then $\mathfrak{M}$ is of the form $\mathfrak{M}(B)$ trivially. If $\mathfrak{M}$ contains a nonzero element, so does $\mathcal{B}$. Let $(f_i(z))_{i\in I}$ be an orthonormal basis for $\mathcal{B}$. Since the dimension of $\mathcal{B}$ is no more than the dimension of $\mathcal{C}$, there is an orthonormal set $(c_i)_{i\in I}$ in $\mathcal{C}$ of the same cardinality. Define a formal power series $B(z)$ with operator coefficients by

$$B(z)c = \sum \bar{c}_i c f_i(z).$$

It is easy to see that $B(z)$ satisfies (1) with $N(B)$ equal to the orthogonal complement of the c_i $(i \in I)$, and that $\mathfrak{M}(B) \subseteq \mathfrak{M}$.

To complete the proof, we will show that the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}(B)$ in $\mathfrak{M}$ is zero. Notice that $\mathcal{B} \subseteq \mathfrak{M}(B)$ and

$$\mathfrak{N} = \mathfrak{M}(B)^\perp \cap \mathfrak{M} \subseteq \mathcal{B}^\perp \cap \mathfrak{M}.$$

Let $f(z)$ be any element of $\mathfrak{N}$. Then $f(z)$ is in $\mathfrak{M}$ and it is orthogonal to $\mathcal{B}$. By the definition of $\mathcal{B}$, $f(z) = zf_1(z)$ for some $f_1(z)$ in $\mathfrak{M}$. If $g(z)$ is any element of $\mathfrak{M}(B)$,

$$\langle f_1(z), g(z)\rangle = \langle zf_1(z), zg(z)\rangle = \langle f(z), g(z)\rangle = 0,$$

since $\mathfrak{M}(B)$ is an ideal. Therefore, $f_1(z)$ is in $\mathfrak{N}$. Continuing by induction, we see that for every $r = 1, 2, 3, \cdots$, there is an $f_r(z)$ in $\mathfrak{N}$ such that $f(z) = z^r f_r(z)$. It follows that $f(z) = 0$ identically. Therefore $\mathfrak{N} = (0)$ and $\mathfrak{M} = \mathfrak{M}(B)$ as asserted.

Added in proof. The methods of this paper have since been used to obtain the existence of invariant subspaces for transformations T which are bounded by 1, when $1 - T^*T$ is completely continuous.

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