ON THE ZEROS OF A CONFLUENT HYPERGEOMETRIC FUNCTION

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1. Introduction. In this note, by using an integral representation due to Erdélyi, we prove a theorem concerning the zeros of the confluent hypergeometric function $\Psi(a, c \mid z)$. As a corollary, we prove that the zeros of the Bessel polynomials for $\delta \geq 0, n \geq 1$, lie in the left-half plane.

2. Fundamental theorems.

**Theorem.** Let $a$ and $c$ be real and $2a - c > -1$. Then any zero $z_s$ of $\Psi(a, c \mid z)$ must satisfy $\text{Re}(z_s) < 0$.

**Proof.** In the formula [2, p. 287, equation (21)]

$$\Gamma(\gamma) \Psi(a, c \mid z) \Psi(a', c \mid y)$$

$$= \int_0^\infty e^{-t} t^{\gamma-1} (x + t)^{-a} (y + t)^{-a'} \binom{a}{y, (x + t)(y + t)} dt,$$

$$\gamma = a + a' - c + 1, \quad \text{Re}(\gamma) > 0, \quad xy \neq 0,$$

let $x = \sigma + i\tau = z$, $y = \sigma - i\tau = \bar{z}$, $a = a'$, $\gamma = 2a - c + 1$. Then

$$\Psi(a, c \mid z) \Psi(a, c \mid \bar{z})$$

$$= \int_0^\infty e^{-t} t^{\gamma-1} [t + \sigma]^2 + \tau^2\binom{a, a}{\gamma, (t + 2\sigma)} dt.$$

Since $\gamma > 0$, the integrand of (2) is a series of strictly positive terms. Thus we have

$$\Psi(a, c \mid z) \Psi(a, c \mid \bar{z}) > 0, \quad \text{Re}(z) \geq 0, \quad z \neq 0.$$

A known result [6] excludes the point $z = 0$ from consideration, and the proof of the theorem is complete.

We now define the Bessel polynomials by

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where

\[(n + \delta)_k = (n + \delta)(n + \delta + 1) \cdots (n + \delta + k - 1),\]

\[(n + \delta)_0 = 1.\]

For a treatment of these polynomials and specializations, see, for example, [1], [3], [4].

McCarthy [5] has proved that for \(n\) fixed and \(\delta\) sufficiently large, the zeros of (4) are in the left-half plane. Applying our theorem, we may prove a more general result.

**Corollary.** Every zero \(z,^{(n)}\) of (4) satisfies \(\text{Re}(z,^{(n)}) < 0\).

**Proof.** Consider the function

\[f(z) = \Psi(-n, -2n - \delta + 1 \mid z).\]

We also have

\[f(z) = z^{2n+\delta} \Psi(n + \delta, 2n + \delta + 1 \mid z).\]

We distinguish two cases.\(^2\) First let \(\delta\) be nonintegral. Then by \([2, \text{p. 257, equation (7)}]\) and (6),

\[f(z) = (n + \delta) \sum_{k=0}^{n} \binom{n}{k} \frac{(-z)^k}{(-2n - \delta + 1)_k},\]

\[= P_n^{(4)}(z),\]

as can be seen by turning the sum around. Now let \(\delta\) be integral and use (7) and \([2, \text{p. 261, equation (13)}]\).

\[f(z) = (n + \delta) \sum_{k=0}^{2n+\delta-1} \binom{n}{k} \frac{(-z)^k}{(-2n - \delta + 1)_k},\]

\[= P_n^{(4)}(z).\]

Our theorem furnishes the desired result for \(\delta > 0\), with \(a = -n, c = -2n - \delta + 1, \text{ and } \gamma = \delta\). Also, we may let \(\delta \to 0\) and the integral (2) in the limit exists, because

\[\text{This is necessary because } \Psi(a, c \mid x) \text{ assumes two different forms depending on whether or not } c \text{ is an integer. Note also (4) defines a polynomial of degree } n, n \geq 0, \delta \neq -1, -2, \cdots, \text{ but we prefer to define } P_n^{(4)}(z) \text{ so the corollary holds.}\]
\[ f(z) = \lim_{\delta \to 0} \int_{0}^{\infty} e^{-\frac{t}{\delta}} \frac{\Gamma(\delta)}{(t + \sigma)^n} \left[ (t + \sigma)^2 + \tau^2 \right]^n \, dt \]
\[ + \int_{0}^{\infty} e^{-t} \left[ (t + \sigma)^2 + \tau^2 \right] \sum_{k=1}^{n} \frac{t^{k-1}(-n)_k}{k! \Gamma(k)} \left[ \frac{(t + 2\sigma)}{(t + \sigma)^2 + \tau^2} \right]^k \, dt \]
\[ = (\sigma^2 + \tau^2)^n + \int_{0}^{\infty} e^{-t} \left[ (t + \sigma)^2 + \tau^2 \right] \sum_{k=1}^{n} \frac{t^{k-1}(-n)_k}{k! \Gamma(k)} \left[ \frac{(t + 2\sigma)}{(t + \sigma)^2 + \tau^2} \right]^k \, dt. \]

Thus the same argument applies in the limiting case and the proof of the corollary is complete.

*Added in proof.* We have noted that Buchholz (Die konfluente hypergeometrische Funktion, p. 90, Springer, Berlin, 1953) proves a similar theorem (excluding \( z = 0 \)) for the related function \( W_{k,m}(z) \).

**REFERENCES**


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