ON THE ZEROS OF A CONFLUENT HYPERGEOMETRIC FUNCTION

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1. Introduction. In this note, by using an integral representation due to Erdélyi, we prove a theorem concerning the zeros of the confluent hypergeometric function \( \Psi(a, c | z) \). As a corollary, we prove that the zeros of the Bessel polynomials for \( \delta \geq 0, n \geq 1 \), lie in the left-half plane.

2. Fundamental theorems.

Theorem. Let \( a \) and \( c \) be real and \( 2a - c > -1 \). Then any zero \( z \) of \( \Psi(a, c | z) \) must satisfy \( \Re(z) < 0 \).

Proof. In the formula [2, p. 287, equation (21)]

\[
\Gamma(\gamma)\Psi(a, c | z)\Psi(a', c | y) = \int_0^\infty e^{-t(\gamma-1)(x+t)^{-a}(y+t)^{-a'}}_{2F1}\left(\begin{array}{c}a, a' \\ \gamma \end{array} \right| \frac{t(x+y+t)}{(x+t)(y+t)} \right) dt, \\
\gamma = a + a' - c + 1, \quad \Re(\gamma) > 0, xy \neq 0,
\]

let \( x = \sigma + i\tau = z, \quad y = \sigma - i\tau = \bar{z}, \quad a = a', \quad \gamma = 2a - c + 1 \). Then

\[
\Psi(a, c | z)\Psi(a, c | \bar{z})
\]

\[
= \int_0^\infty \frac{e^{-t(\gamma-1)[(t+\sigma)^2 + \tau^2]^{-a}}_{2F1}\left(\begin{array}{c}a, a' \\ \gamma \end{array} \right| \frac{t(t+2\sigma)}{(t+\sigma)^2 + \tau^2} \right) dt.
\]

Since \( \gamma > 0 \), the integrand of (2) is a series of strictly positive terms. Thus we have

\[
\Psi(a, c | z)\Psi(a, c | \bar{z}) > 0, \quad \Re(z) \geq 0, \quad z \neq 0.
\]

A known result [6] excludes the point \( z = 0 \) from consideration, and the proof of the theorem is complete.

We now define the Bessel polynomials by

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\[ P_n^{(\delta)}(z) = \sum_{k=0}^{n} \binom{n}{k} (n + \delta)_k z^{n-k}, \quad \delta \geq 0, \ n \geq 1, \]

where

\[(n + \delta)_k = (n + \delta)(n + \delta + 1) \cdots (n + \delta + k - 1), \]

\[(n + \delta)_0 = 1. \]

For a treatment of these polynomials and specializations, see, for example, [1], [3], [4].

McCarthy [5] has proved that for \( n \) fixed and \( \delta \) sufficiently large, the zeros of (4) are in the left-half plane. Applying our theorem, we may prove a more general result.

**Corollary.** Every zero \( z_{n,(\delta)} \) of (4) satisfies \( \Re(z_{n,(\delta)}) < 0. \)

**Proof.** Consider the function

\[ f(z) = \Psi(-n, -2n - \delta + 1 | z). \]

We also have

\[ f(z) = z^{2n+\delta+1} \Psi(n + \delta, 2n + \delta + 1 | z). \]

We distinguish two cases.\(^2\)

- First let \( \delta \) be nonintegral. Then by [2, p. 257, equation (7)] and (6),

\[ f(z) = (n + \delta) \sum_{k=0}^{n} \binom{n}{k} \frac{(-z)^k}{(-2n - \delta + 1)_k} \]

\[ = P_n^{(\delta)}(z), \]

as can be seen by turning the sum around. Now let \( \delta \) be integral and use (7) and [2, p. 261, equation (13)].

\[ f(z) = (n + \delta)_n \sum_{k=0}^{2n+\delta-1} \binom{n}{k} \frac{(-z)^k}{(-2n - \delta + 1)_k}, \]

\[ = P_n^{(\delta)}(z). \]

Our theorem furnishes the desired result for \( \delta > 0 \), with \( a = -n \), \( c = -2n - \delta + 1 \), and \( \gamma = \delta \). Also, we may let \( \delta \to 0 \) and the integral (2) in the limit exists, because

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\(^2\) This is necessary because \( \Psi(a, c | x) \) assumes two different forms depending on whether or not \( c \) is an integer. Note also (4) defines a polynomial of degree \( n \), \( n \geq 0 \), \( \delta \neq -1, -2, \cdots \), but we prefer to define \( P_n^{(\delta)}(z) \) so the corollary holds.
\[ f(z) = \lim_{\delta \to 0} \int_0^\infty \frac{e^{-it\delta-1}}{\Gamma(\delta)} [(t + \sigma)^2 + \tau^2] e^{-it[(t + \sigma)^2 + \tau^2]} \sum_{k=1}^\infty \frac{t^{k-1}(-n)^k}{k! \Gamma(k)} dt \]

\[ = \left( (2\sigma + \tau)^n + \int_0^\infty e^{-it[(t + \sigma)^2 + \tau^2]} \sum_{k=1}^\infty \frac{t^{k-1}(-n)^k}{k! \Gamma(k)} \right) e^{-(t + \sigma)^2 + \tau^2} \]

Thus the same argument applies in the limiting case and the proof of the corollary is complete.

*Added in proof.* We have noted that Buchholz (Die konfluente hypergeometrische Funktion, p. 90, Springer, Berlin, 1953) proves a similar theorem (excluding \( z = 0 \)) for the related function \( W_{k,m}(z) \).

**References**