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**A REPRESENTATION THEOREM FOR CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES**

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Let $E^n$ denote the $n$-dimensional unit cube in Euclidean space; designate the closed unit interval, $[0, 1]$, by $E$. We prove in this note the following

**Theorem.** For any natural number $n$, $n \geq 2$, there exist real, monotonic increasing functions, $h^p(x)$, $1 \leq p \leq n$, dependent on $n$, and having the following properties:

(i) The function

$$
\sum_{1 \leq p \leq n} h^p(x_p)
$$

separates all points of $E^n$:

$$
\sum_{1 \leq p \leq n} h^p(x_p) \neq \sum_{1 \leq p \leq n} h^p(y_p)
$$

unless $x_p = y_p$ for all admitted values of $p$.

(ii) Every continuous function of $n$ variables, $f(x_1, \cdots, x_n)$, with domain $E^n$, can be represented in the form

$$
f(x_1, \cdots, x_n) = g \left[ \sum_{1 \leq p \leq n} h^p(x_p) \right].
$$

Clearly, the function $g$ will, in general, be discontinuous.

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As shown by V. I. Arnol'd [1], even a simple function such as \( f(x, y) = xy \) cannot be represented in the form (1) if the functions \( g \) and \( h^p \) are required to be continuous.

**Proof.** We represent the real numbers in the interval \( E \) to the base \( n \):

\[
(2) \quad x = \sum_{1 \leq s < n} s_n \cdot n^{-s},
\]

where \( s_n \) is an index with domain \( 0 \leq s_n \leq n - 1 \) for each \( v \). To have a one-to-one correspondence between the real numbers, \( x \in E \), and the infinite series (2), we normalize (2) by requiring that for no \( N > 1 \) is \( s_n = n - 1 \) for all \( v \geq N \), except when \( s_n = n - 1 \) for all \( v \).

To prove (i), we set for each admitted \( p \)

\[
(3) \quad x_p = \sum_{1 \leq s < n} s_{pv} \cdot n^{-s},
\]

where \( 0 \leq s_{pv} \leq n - 1 \), the \( s_{pv} \) being subject to the normalizing restriction just described, and define the functions

\[
(4) \quad h^p(x_p) = \sum_{1 \leq s < n} s_{pv} \cdot n^{-n - p + 1}.
\]

Clearly, (4) is a representation of real numbers in \( E \) to the base \( n^p \). By their construction, the functions \( h^p(x) \) are monotonic increasing and bounded for \( x \in E \) (and hence, continuous almost everywhere).

Let \( p \) be fixed; consider an infinite sequence \( \{s_{pv}\} \); this sequence determines simultaneously unique numbers \( x_p \) and \( h^p(x_p) \). It follows that the correspondence

\[
(5) \quad x_p \leftrightarrow h^p(x_p)
\]

is one-to-one for each admitted \( p \).

Let us write

\[
(6) \quad \sum_{1 \leq s < n} h^p(x_p) = n^{1-n} \sum_{1 \leq s < n} t_s \cdot n^{-n},
\]

where

\[
(7) \quad t_s = \sum_{1 \leq s < n} s_{pv} \cdot n^{n-p}.
\]

We first show that the normalizing restriction imposed on the \( s_{pv} \) carries with it the analogous restriction for the \( t_s \), proving thereby
that the right side of (6) is a unique representation of real numbers to the base $n^n$. That is, we demonstrate that $t \prec n^n - 1$ for infinitely many $v$, unless $t, = n^n - 1$ for all $v$.

The specified domain of the $s_{pv}$ is $0 \leq s_{pv} \leq n - 1$; accordingly we have

\begin{equation}
0 \leq \sum_{1 \leq p \leq n} s_{pv} \cdot n^{-p} \leq (n - 1) \sum_{1 \leq p \leq n} n^{-p} = n^n - 1.
\end{equation}

Since $s_{pv} < n - 1$ for infinitely many $v$,

$$t_\nu = \sum_{1 \leq p \leq n} s_{pv} \cdot n^{-p} < n^n - 1$$

for infinitely many $\nu$, unless $s_{pv} = n - 1$ for all values of $p$ and $\nu$.

To complete the proof of (i) it remains, therefore, only to show that the correspondence

\begin{equation}
(x_1, \ldots, x_n) \rightarrow \sum_{1 \leq p \leq n} h^p(x_p)
\end{equation}

is one-to-one. We demonstrate, namely, that the right side of (6) determines a unique point in $E^n$:

Let

$$\sum_{1 \leq p \leq n} h^p(y_p) = n^{1-n} \sum_{1 \leq r \leq \infty} t'_r \cdot n^{-nr};$$

if

\begin{equation}
\sum_{1 \leq p \leq n} h^p(x_p) = \sum_{1 \leq p \leq n} h^p(y_p),
\end{equation}

then $t_\nu = t'_\nu$ for all values of $\nu$.

By the definition of the summands in (10), this equation is equivalent to the statement that

$$t_1 - t'_1 = \sum_{2 \leq r \leq \infty} (t'_r - t_r) \cdot n^{-nr} = \alpha,$$

where $|\alpha| \leq 1$, as shown with a simple calculation. That this inequality is strict follows from the fact that $|\alpha| = 1$ if and only if $|t'_r - t_r| = n^n - 1$ for all $r \geq 2$, and then, according to the normalizing restriction, $|t'_r - t_r| = n^n - 1 = 1$. The last equality is clearly impossible.

We now prove the assertion made, that $t_\nu = t'_\nu$ for all $\nu$, by induction on $\nu$. Since $|\alpha| < 1$ and the difference $|t'_r - t_r|$ is integral or zero,
it follows that $\alpha = 0$ and, hence, $t_i = t'_i$. Suppose now that $t_i = t'_i$ for all $i \leq k$, where $k \geq 1$. Then equation (10) is equivalent to the statement that

$$t_{k+1} - t'_{k+1} = \sum_{k+2 \leq p \leq \infty} (t'_i - t_i) \cdot n^{n(k+1-r)} = \alpha',$$

where $|\alpha'| \leq 1$. The above reasoning shows that inevitably $t_{k+1} = t'_{k+1}$, thereby completing the induction.

Now let

$$t'_i = \sum_{1 \leq p \leq n} s'_{ip} \cdot n^{n-p};$$

the relation $t_i = t'_i$ permits us to write

$$0 \leq s_{nr} = s'_{nr} + \left[ n \cdot \sum_{1 \leq p \leq n-1} (s'_{ip} - s_{ip}) \cdot n^{n-p-1} \right] \leq n - 1.$$  

Since $s'_{ip}$ is non-negative, and the expression in brackets is an integral multiple of $n$, this is impossible, unless $s_{ip} = s'_{ip}$ for all admitted values of $p$. This shows that $t_i \neq t'_i$ unless $s_{ip} = s'_{ip}$ for all $p$, and hence the correspondence (9) is one-to-one.

The correspondence (9) maps the unit cube, $E^n$, onto the closed interval $[0, n^{1-n}]$, in a one-to-one manner. To each point

$$y = \sum_{i \leq p \leq n} h(x_p)$$

in this interval, the assignment $g(y) = f(x_1, \ldots, x_n)$, therefore, is defined uniquely. The function $g$ is that demanded in part (ii) of our theorem.

**Bibliography**

1. V. I. Arnol'd, *On the representability of a function of two variables in the form $\chi[\phi(x) + \psi(y)]$*, Uspehi Mat. Nauk. 12 (1957), no. 2(74), 119–121. (Russian)