A NEW PROOF OF DEICKE'S THEOREM ON 
HOMOGENEOUS FUNCTIONS 

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We denote by \( R_n \) the \( n \)-dimensional number space of points \( \{ x^1, x^2, \ldots, x^n \} \), where the \( x^i \) are real numbers, and we use \( R_n' \) to denote \( R_n \) with the point \( \{ 0, 0, \ldots, 0 \} \) removed. Let \( L \) be a positive function of class \( C^4 \) defined on \( R_n' \) and positively homogeneous of degree one. Then, introducing the matrix \( g \) of elements

\[
g_{ij} = \frac{\partial^2 (\frac{1}{2} L^2)}{\partial x^i \partial x^j},
\]

we give a new proof of the following theorem, due originally to A. Deicke [1].

**Theorem.** Let \( \det g \) be constant on \( R_n' \). Then \( g \) is constant on \( R_n' \).

It is known that the assumptions made imply that the matrix \( g \) is positive definite [1]. We first prove

**Lemma 1.** Let \( x, y \) be any two points in \( R_n' \). Then \( \text{Tr} g^{-1}(x)g(y) \geq n \).

**Proof.** Since the matrices \( g(x), g(y) \) are positive definite, the characteristic roots of \( g(y) \) with respect to \( g(x) \) are all positive. These roots are also the characteristic roots of the matrix \( g^{-1}(x)g(y) \) so that, using the inequality between arithmetic and geometric means,

\[
\text{Tr} g^{-1}(x)g(y) \geq n (\det g^{-1}(x)g(y))^{1/n} = n.
\]

We next introduce the elliptic differential operator

\[
\Delta = \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j},
\]

where \( g^{ij} \) denotes the general element of the matrix \( g^{-1}(x) \) and prove

**Lemma 2.** The matrix \( \Delta g \) is positive semi-definite.

**Proof.** Define a function \( \phi \) by \( \phi(x) = \text{Tr} g^{-1}(x)g(y) \). Since \( \phi(x) = n \), Lemma 1 shows that \( \phi \) has a minimum at \( y = x \) and hence the matrix of elements

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is positive semi-definite for \( y = x \). This matrix is also equal to \( \Delta g \) for \( y = x \).

We complete the proof of the theorem by using a theorem due to E. Hopf [2, Theorem 2.1]. Lemma 2 implies that, for each \( h, \Delta g_{hk} \geq 0 \). Since \( g_{hk} \) is positively homogeneous of degree zero and hence attains a maximum on \( R_n' \), Hopf's theorem shows that \( g_{hk} \) is constant on \( R_n' \). Lemma 2 now implies that \( \Delta g_{hk} = 0 \) for all \( h, k \) and, as before, Hopf's theorem shows that \( g_{hk} \) is constant on \( R_n' \).

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REFERENCES

1. A. Deicke, Über die Finsler-Raume mit \( A_1 = 0 \), Arch. Math. 4 (1953), 45-51.