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ON CLASSES OF UNIVALENT CONTINUED FRACTIONS

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1. **Introduction.** From results of Leighton and Scott [3], there is a unique one-to-one correspondence between formal power series $w^{-1}+\sum_{n=2}^{\infty} c_n w^{-n}$ and C-fractions

$$(1.1)\qquad F(w) = \frac{1}{w} - \frac{a_1}{w^{\delta_1}} - \frac{a_2}{w^{\delta_2}} - \cdots - \frac{a_n}{w^{\delta_n}} - \cdots,$$

where δ_n is an integer, $\delta_1 \geqq 0$, $\delta_{n+1}+\delta_n \geqq 1$, and $a_{n+p}=0$ whenever $a_p=0$ for $n=1, 2, \cdots$. For a fixed continued fraction (1.1), let K_F denote the class of formal power series which correspond to C-fractions of the form

$$(1.2)\qquad \frac{1}{w} - \frac{a_1'}{w^{\delta_1}} - \frac{a_2'}{w^{\delta_2}} - \cdots - \frac{a_n'}{w^{\delta_n}} - \cdots,$$

where $|a_n'| \leqq |a_n|$, $n=1, 2, \cdots$. In order that each power series in K_F represent an analytic function in $|w| \geqq 1$ it is necessary and sufficient that $|a_n| \leqq g_n(1-g_{n-1})$, where $0<g_{n-1}\leqq 1$, $n=1, 2, \cdots$, and $g_{p-1}=1$ if and only if $a_p=0$ [2, p. 374]. Conditions on the parameters g_n of the chain sequence $\{g_n(1-g_{n-1})\}_{n=1}^{\infty}$ which imply that each

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function in K_F is univalent in $|w|>1$ are determined in §2. A similar problem for a class of analytic functions with certain J-fraction expansions is also considered. In §3 conditions on the parameters which imply that each function in K_F is starlike with respect to the origin in $|w|>1$ are given. These results extend those of Thale [7], Perron [6], Merkes and Scott [5] which, in our terminology, treat the class K_F where F is given by (1.1) with $a_n=a\neq 0$, $\delta_{2n-1}=\delta$, $\delta_{2n}=1$, δ a nonnegative integer, $n=1, 2, \cdots$.

2. **Univalence.** Let $F(w)$ be a fixed C-fraction (1.1) and let K_F be the associated class of formal power series.

THEOREM 2.1. *Each $f(w)\in K_F$ is analytic and univalent in $|w|>1$ if $\{|a_j|\}_{j=1}^{\infty}$ is a chain sequence with parameters g_j, $0<g_j\leqq 1$, $j=0, 1, 2, \cdots$, and*

$$\Lambda = \sum_{j=1}^{\infty} |\delta_j| \prod_{p=1}^{j} \frac{1-g_{p-1}}{g_p} \leqq 1. \tag{2.1}$$

Furthermore, if $\delta_j\geqq 0$, $j=1, 2, \cdots$, and $\{g_j\}$ is the maximal parameter sequence [8, p. 81], there is a function in K_F which is not univalent in $|w|>R$ for any $R<1$ whenever $\Lambda=1$.

PROOF. For each nonnegative integer n, define

$$f_n(w) = \frac{1}{w^{\delta_n}} - \frac{a'_{n+1}}{w^{\delta_{n+1}}} - \frac{a'_{n+2}}{w^{\delta_{n+2}}} - \cdots, \tag{2.2}$$

where $\delta_0=1$ and $f_0(w)\in K_F$. Since $|a'_j|\leqq|a_j|$, where $\{|a_j|\}$ in (1.1) is a chain sequence, each $f_n(w)$ is analytic in $|w|\geqq 1$ [2] and

$$0 < |f_n(w)w^{\delta_n}| \leqq 1/g_n \tag{2.3}$$

for $|w|\geqq 1$ [8, p. 46]. For a fixed w_1 in $|w|\geqq 1$, it is easily shown by (2.2) that

$$f_0(w)-f_0(w_1) = -\sum_{j=0}^{m} (w^{\delta_j}-w_1^{\delta_j}) \prod_{p=0}^{j} a'_p f_p(w) f_p(w_1) + R_m, \tag{2.4}$$

where $a'_0=1=\delta_0$ and

$$R_m = a'_{m+1}[f_{m+1}(w)-f_{m+1}(w_1)] \prod_{p=0}^{m} a'_p f_p(w_1) f_p(w).$$

By (2.3) and the fact that $|a'_j|\leqq g_j(1-g_{j-1})$, $j=1, 2, \cdots$, we obtain, for $|w|\geqq 1$,

$$|R_m| \leqq [|w|^{-k_{m+1}}|w_1|^{-k_m} + |w|^{-k_m}|w_1|^{-k_{m+1}}] \frac{g_{m+1}}{g_0^2} \prod_{p=1}^{m+1} \frac{1-g_{p-1}}{g_p}$$
$$\leqq \frac{2}{g_0^2} \prod_{p=1}^{m+1} \frac{1-g_{p-1}}{g_p},$$

where $k_n = \delta_0 + \delta_1 + \cdots + \delta_n > 0$ by the conditions $\delta_0 = 1$, $\delta_{n-1} + \delta_n \geqq 1$, $n = 1, 2, \cdots$. The convergence of the series Λ in (2.1) now implies $R_m \to 0$ uniformly as $m \to \infty$ for $|w| \geqq 1$. Hence, by (2.4),

$$(2.5) \qquad f_0(w) - f_0(w_1) = -\sum_{j=0}^{\infty} (w^{\delta_j} - w_1^{\delta_j}) \prod_{p=0}^{j} a_p' f_p(w) f_p(w_1)$$

and

$$(2.6) \qquad f_0'(w) = -\sum_{j=0}^{\infty} \delta_j w^{\delta_j - 1} \prod_{p=0}^{j} a_p' f_p^2(w), \qquad |w| \geqq 1.$$

Now for $|w| \geqq 1$, $|w_1| \geqq 1$,

$$|w^{-k_{j-1}} w_1^{-k_j} - w^{-k_j} w_1^{-k_{j-1}}| \leqq |w - w_1| |\delta_j|,$$

where $k_j = \delta_0 + \cdots + \delta_j > 0$, $j = 0, 1, 2, \cdots$. In conjunction with (2.3), this shows by (2.5) that

$$(2.7) \qquad \left| \frac{f_0(w) - f_0(w_1)}{(w - w_1) f_0(w) f_0(w_1)} + 1 \right| \leqq \Lambda,$$

where Λ, given by (2.1), does not exceed unity. Since the function within the absolute-value symbol, when appropriately defined at $w = w_1$, is analytic in $|w| \geqq 1$, the maximum modulus principle gives strict inequality for $|w| > 1$ provided $f_0(w) \not\equiv w^{-1}$. In particular, $f_0(w) \neq f_0(w_1)$ for $w \neq w_1$, $|w| > 1$, $|w_1| > 1$. This proves the first part of the theorem.

Let

$$F_0(w) = \frac{1}{w} \, \frac{}{+} \, \frac{|a_1|}{w^{\delta_1}} \, \frac{}{-} \, \frac{|a_2|}{w^{\delta_2}} \, \frac{}{-} \, \cdots,$$

which is in K_F, and let

$$F_j(w) = \frac{1}{w^{\delta_j}} \, \frac{}{-} \, \frac{|a_{j+1}|}{w^{\delta_{j+1}}} \, \frac{}{-} \, \cdots, \qquad j = 1, 2, \cdots.$$

When $\{g_n\}$ is the maximal parameter sequence of $\{|a_n|\}$, $F_j(1) = 1/g_j$, $j = 1, 2, \cdots$ [8, p. 81]. Since $|a_j| = g_j(1 - g_{j-1})$, $j = 1, 2, \cdots$, it follows from (2.6) that, for $\delta_j \geqq 0$,

$$\frac{F_0'(1)}{[F_0(1)]^2} = -1 + \sum_{j=1}^{\infty} \delta_j \prod_{p=1}^{j} \frac{1 - g_{p-1}}{g_p} = -1 + \Lambda.$$

Therefore, if $\Lambda = 1$, $F_0(w)$ is not univalent in $|w| > R$ for any $R < 1$.

If a formal power series $f(w)$ corresponds to a J-fraction, it is often possible to obtain a larger region of analyticity and univalence for $f(w)$ than that obtained from Theorem 2.1. A result of this kind is given by the following:

THEOREM 2.2. *Let b_n and $\beta_n > 0$ be complex numbers, $n = 1, 2, \cdots$. Let D be a region such that $w \in D$ implies $|w - b_n| > \beta_n$, $n = 1, 2, \cdots$. The J-fraction*

$$\frac{1}{w - b_1 -} \frac{a_1^2}{w - b_2 -} \cdots \frac{a_n^2}{-\, w - b_{n+1} -} \cdots \tag{2.8}$$

represents an analytic univalent function in D provided $\{|a_n^2|/\beta_n\beta_{n-1}\}_{n=1}^{\infty}$ is a chain sequence with parameters g_n, $0 < g_n \leqq 1$, $n = 0, 1, \cdots$, such that

$$L = \sum_{n=1}^{\infty} \frac{\beta_1}{\beta_{n+1}} \prod_{p=1}^{n} \frac{1 - g_{p-1}}{g_p} \leqq 1. \tag{2.9}$$

The proof is similar to that of Theorem 2.1 and therefore is omitted.

In particular, when $|a_n| \leqq N/3$, $\beta_n \geqq \beta_1 > 0$, $n = 1, 2, \cdots$, the choice $g_n = 2/3$, $n = 0, 1, \cdots$, shows (2.8) is analytic and univalent in any region contained in the common part of the regions $|w - b_n| > \sqrt{(2)}N/2$. This statement includes Thale's results on J-fractions [7]. Indeed, if $|b_n| \leqq M/3$, $n = 1, 2, \cdots$, then a domain of univalence of (2.8) is $|w| > (3\sqrt{(2)}N + 2M)/6$. Moreover, if $\operatorname{Im} b_n \leqq 0$, $n = 1, 2, \cdots$, which is the case whenever (2.8) is positive definite, then $\operatorname{Im} w > \sqrt{(2)}N/2$ is a domain of univalence of the J-fraction. Each of these results is sharp. For, by a suitable choice of the real number a, the function

$$\frac{2}{3(w-a) - \sqrt{((w-a)^2 + 4N^2/9)}} = \frac{1}{w - a -} \frac{N^2/9}{w - a +} \frac{N^2/9}{w - a} + \frac{N^2/9}{w - a} + \cdots,$$

whose derivative vanishes at $w = a + \sqrt{(2)}Ni/2$, is not univalent in a given open region which properly contains one of these domains of univalence.

3. **Starlikeness.** As a simple consequence of (2.7) we have the following sufficient condition for each power series in K_F to represent a starlike function.

THEOREM 3.1. *Each* $f(w)\in K_F$ *is analytic, univalent, and starlike with respect to the origin in* $|w|>1$ *if* $\{|a_j|\}_{j=1}^{\infty}$ *is a chain sequence with parameters* g_j, $0<g_j\leqq 1$, *and*

$$\Lambda \leqq [g_0(2-g_0)]^{1/2},$$

where Λ *is defined in* (2.1).

PROOF. Let $f(w)\in K_F$ correspond to the C-fraction (1.2). From (2.7), we obtain

$$\left|\frac{wf'(w)}{f(w)}+wf(w)\right| \leqq |wf(w)|\,\Lambda.$$

This implies $\operatorname{Re}\{wf'(w)/f(w)\}\leqq 0$ for $|w|>1$, which is sufficient for starlikeness with respect to the origin, provided

$$\cos\theta \geqq \Lambda, \qquad \theta = \arg\{wf(w)\}. \tag{3.1}$$

Since [8, p. 46]

$$\left|wf(w)-\frac{1}{g_0(2-g_0)}\right| \leqq \frac{1-g_0}{g_0(2-g_0)}, \qquad |w|>1,$$

it follows that

$$\cos\theta \geqq [g_0(2-g_0)]^{1/2}.$$

In conjunction with (3.1) this establishes the theorem.

In general, it is conjectured that, for any C-fraction (1.1) such that the hypothesis of Theorem 2.1 hold, each $f(w)\in K_F$ is analytic, univalent, and starlike with respect to the origin in $|w|>1$.

Let, in particular, $F(w)$ be the C-fraction (1.1) with $a_n=g_0(1-g_0)$, $1/2\leqq g_0<1$, $\delta_{2n-1}=0$, $\delta_{2n}=1$, $n=1, 2, \cdots$. By Theorem 3.1, each $f(w)\in K_F$ is starlike with respect to the origin if $g_0>.60$. This is an improvement over the previously obtained [5] lower estimate of $(3-\sqrt{(3)})/2$. Recently, F. V. Atkinson [1] has verified a conjecture of the authors that the radius of starlikeness is the radius of univalence in this case and in the slightly more general situation treated in [5].

An argument similar to that of Theorem 3.1 gives the following result on starlikeness of J-fractions.

THEOREM 3.2. *Let b_n and $\beta_n > 0$, $n = 1, 2, \cdots$, be complex numbers such that $|w - b_n| > \beta_n$, $n = 1, 2, \cdots$, whenever $|w| > R$. The J-fraction* (5.8) *is analytic, univalent, and starlike with respect to the origin in $|w| > R$ provided $b_1 = 0$ and $\{|a_n|^2/\beta_n\beta_{n+1}\}_{n=1}^{\infty}$ is a chain sequence with parameters g_n, $0 < g_n \leqq 1$, and*

$$[g_0(2 - g_0)]^{1/2} \geqq L,$$

where L is given by (2.9).

A similar result for the case $b_1 \neq 0$ can be deduced but because of its complexity is omitted. This result slightly improves those in [4] and lends credence to the conjecture that the radius of starlikeness is the same as the radius of univalence for bounded J-fractions.

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