THE MAXIMUM TERM OF AN ENTIRE SERIES WITH GAPS

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Let $\sum a_p z^p$ denote the power series for an entire function of order $\rho$ and lower order $\lambda$. S. M. Shah [2] has shown that

$$\lim \inf \left[ \mu(r) \right]^{1/s(r)} \leq e^{1/\rho},$$

$$\lim \sup \left[ \mu(r) \right]^{1/s(r)} \geq e^{1/\lambda},$$

where $\mu(r)$ denotes the maximum term of $\sum|a_p| r^p$ and $v(r)$ is the largest integer $p$ for which $\mu(r) = |a_p| r^p$.

The object of the present note is to obtain a sharper form of (1) for those entire series which possess Hadamard gaps. For this purpose let the subsequence $\{a_p\}$ contain all the nonvanishing terms of $\{a_p\}$, and suppose that

$$\lim \inf \frac{p_{m+1}}{p_m} \geq 1 + \theta > 1.$$  \hspace{1cm} (2)

We shall prove the following

**Theorem.** Suppose $\sum a_p z^{p_m}$ is an entire series of order $\rho$ and lower order $\lambda$ whose gaps satisfy (2). Then

$$\lim \inf \left[ \mu(r) \right]^{1/s(r)} \leq \alpha^{1/\rho},$$

$$\lim \sup \left[ \mu(r) \right]^{1/s(r)} \geq \beta^{1/\lambda},$$

where

$$\alpha = (1 + \theta)^{1/\theta}$$

and

$$\beta = (1 + \theta)^{(1+\theta)/\theta}.$$  \hspace{1cm} (3)

We call attention to the fact that a series which satisfies (2) need not be of irregular growth; much larger gaps are needed [3] to insure that $\lambda < \rho$.

**Proof.** The function $v(r)$ is a nondecreasing step function which is continuous from the right and assumes only nonnegative integer

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values. Therefore there is a nondecreasing sequence \( \{R_k\} \) which \( \nu(r) \) "counts," i.e.,

\[
\nu(r) = \sum_{R_k \leq r} 1.
\]

For convenience we assume that \( R_1 = 1 \). No generality is lost since this is equivalent to requiring that

\[
|a_0| = \max_{p \geq 1} |a_p|.
\]

For each \( k \geq 1 \), let

\[
(4a) \quad t_k = \log R_k - \frac{\log R_1 + \cdots + \log R_k}{k}
\]

and

\[
(4b) \quad u_k = \log R_{k+1} - \frac{\log R_1 + \cdots + \log R_k}{k}.
\]

In addition to satisfying (2), we assume the sequence

\[
p_0, p_1, \ldots, p_m, \ldots
\]

is such that \( p_0 = 0 \) and \( p_1 = 1 \). For notational convenience we shall always denote \( p_m \) by \( n \).

The following relations are easily verified by examining the local minima and maxima of the quantities involved:

\[
\liminf_{m \to \infty} t_n = \liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r)},
\]

\[
\limsup_{m \to \infty} u_n = \limsup_{r \to \infty} -\frac{\log \mu(r)}{\nu(r)},
\]

\[
\liminf_{m \to \infty} \frac{\log R_n}{\log n} = \liminf_{r \to \infty} \frac{\log r}{\log \nu(r)} = \frac{1}{\rho},
\]

\[
\limsup_{m \to \infty} \frac{\log R_{n+1}}{\log n} = \limsup_{r \to \infty} \frac{\log r}{\log \nu(r)} = \frac{1}{\lambda}.
\]

We shall also need estimates for the quantities

\[
A(n) = 1 + \sum_{j=1}^{m-1} \left[ \frac{p_{j+1}}{p_j} - 1 \right]
\]

and
\[ B(n) = 1 + \sum_{j=1}^{m-1} \left[ 1 - \frac{p_{j+1}}{p_j} \right]. \]

For this purpose let
\[ x_j = \frac{p_{j+1}}{p_j} - 1, \quad j = 1, 2, 3, \ldots. \]

Then
\[ \log n = \sum_{j=1}^{m-1} \log(1 + x_j). \]

From (2) and the fact that \((1/x) \log(1+x)\) is a decreasing function, we obtain
\[ \lim_{m \to \infty} \sup \frac{\log n}{A(n)} \leq \frac{\log(1 + \theta)}{\theta}. \]

A similar argument shows that
\[ \lim_{m \to \infty} \inf \frac{\log n}{B(n)} \geq \frac{(1 + \theta) \log(1 + \theta)}{\theta}. \]

Having taken care of the above preliminaries, we turn now to the main body of the proof. Inverting the systems of equations (4a) and (4b) yields (since \(R_1 = 1\))

(5a) \[ \log R_n = t_n + \sum_{k=2}^{n} \frac{t_k}{k - 1} \]

and

(5b) \[ \log R_{n+1} = u_n + \sum_{k=1}^{n-1} \frac{u_k}{k + 1}. \]

The values assumed by \(\nu(r)\) are all terms of \(\{p_m\}\); therefore
\[ \log R_k = \log R_{p_{j+1}}, \quad p_j < k < p_{j+1}, \]
from which it follows that
\[ t_k = \frac{p_{j+1}}{k} t_{p_{j+1}}, \quad p_j < k \leq p_{j+1}, \]
and
\[ u_k = \frac{p_j}{k} u_{p_j}, \quad p_j \leq k < p_{j+1}. \]
Substituting these expressions in (5a) and (5b), we obtain

\[
\log R_n = t_n + \sum_{j=1}^{m-1} t_{p_{j+1}} \left( \frac{p_{j+1}}{p_j} - 1 \right)
\]

and

\[
\log R_{n+1} = u_n + \sum_{j=1}^{m-1} u_{p_j} \left( 1 - \frac{p_j}{p_{j+1}} \right).
\]

From (6a) and (6b) it follows (cf. [1, p. 52, Theorem 9]) that

\[
\liminf_{m \to \infty} t_n \leq \liminf_{m \to \infty} \frac{\log R_n}{A(n)}
\]

and

\[
\limsup_{m \to \infty} u_n \geq \limsup_{m \to \infty} \frac{\log R_{n+1}}{B(n)}.
\]

From (7a) we have

\[
\liminf_{r \to \infty} \frac{\log \mu(r)}{v(r)} \leq \liminf_{m \to \infty} \frac{\log R_n}{A(n)}
\]

\[
\leq \left[ \liminf_{m \to \infty} \frac{\log R_n}{\log n} \right] \left[ \limsup_{m \to \infty} \frac{\log n}{A(n)} \right]
\]

\[
\leq \frac{\log(1 + \theta)}{\rho \theta}.
\]

Therefore

\[
\liminf_{r \to \infty} [\mu(r)]^{1/v(r)} \leq \alpha^{1/\rho}.
\]

The remaining portion of the theorem follows similarly from (7b).

We note that \(\alpha\) and \(\beta\) tend respectively to 1 and \(\infty\) as \(\theta \to \infty\). In conjunction with our theorem this remark implies the following

**Corollary.** Suppose that \(\sum a_p z^p\) is an entire function of positive finite order, and

\[
\lim_{m \to \infty} \frac{p_{m+1}}{p_m} = \infty.
\]

Then

\[
\liminf_{r \to \infty} [\mu(r)]^{1/v(r)} = 1.
\]
and

\[ \limsup_{r \to \infty} [\mu(r)]^{1/\nu(r)} = \infty. \]

**REFERENCES**


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**MIXED BOUNDARY-VALUE PROBLEMS IN THE PLANE**

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Let $R$ be a region in the plane bounded by a simple analytic curve $C$ composed of $N$ arcs $C_1 \cdots C_N$. Let $a_m, b_m, f_m$ be analytic functions on $C_m$. Suppose $g(x, y)$ is non-negative in $R$. The mixed boundary-value problems discussed here require the determination of a solution of

\[
\begin{align*}
\Delta u - qu &= 0 & \text{in } R, \\
 a_m u_n - b_m u &= f_m & \text{on } C_m,
\end{align*}
\]

$n$ the exterior normal. The problem is called regular if on each $C_m$ either

(i) $a_m > 0, \quad b_m \geq 0$

or

(ii) $a_m \equiv 0, \quad b_m > 0$.

This note presents an existence theorem based on integral equations. The method is an extension of the solution of the Dirichlet problem by simple layers as in [1] and [4]. It is intended also to provide information as to the behavior of $u$ at the ends of the $C_k$.

**Theorem 1.** Every regular mixed problem has a unique solution.

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