AN EXTENSION OF TATE'S THEOREM ON COHOMOLOGICAL TRIVIALITY

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Let $G$ be a finite group and $f: A \rightarrow B$ a homomorphism of $G$-modules. In one form, Tate's theorem says that if, for some $r$ and all subgroups $U$ of $G$, $\hat{H}^{r-1}(U, f)$ is a surjection, $\hat{H}^r(U, f)$ is an isomorphism, and $\hat{H}^{r+1}(U, f)$ is an injection, then $\hat{H}^n(U, f)$ is an isomorphism for all $U$ and all $n$. Whaples has asked if the modification of this theorem stated below is true, and this paper answers Whaples' question affirmatively.

**Theorem.** If, for some integer $r$ and every subgroup $U$ of the finite group $G$, $\hat{H}^r(U, f)$ and $\hat{H}^{r+1}(U, f)$ are isomorphisms, then $\hat{H}^n(U, f)$ is an isomorphism for every $n$ and every subgroup $U$.

**Proof.** By the Sylow subgroup argument in cohomology of finite groups it is sufficient to prove the theorem for $p$-groups. For $p$-groups we proceed by induction. For the trivial group the theorem is clear, so let $G$ be a nontrivial $p$-group and assume the truth of the theorem for $p$-groups of lower order. We prove below that $\hat{H}^n(U, f)$ is an isomorphism for all $U$ and all $n \leq r + 1$. The proof for $n \geq r$ is analogous. By dimension shifting we may assume $r = -3$, that is, that $H_3(U, f)$ and $H_4(U, f)$ are isomorphisms for all $U$. (I mean the ordinary homology groups.) Let $H$ be a maximal subgroup of $G$. We have the following commutative diagram with obvious vertical arrows.

$$
\begin{array}{cccccc}
H_1(G/H, H_1(H, A)) & \rightarrow & K_2(A) & \rightarrow & H_2(G/H, A_H) & \rightarrow & H_1(H, A) \rightarrow H_1(G, A) \rightarrow H_1(G/H, A_H) \rightarrow 0 \\
1 & \downarrow & 1 & \downarrow & 1 & \downarrow & 1 \\
H_1(G/H, H_1(H, B)) & \rightarrow & K_2(B) & \rightarrow & H_2(G/H, B_H) & \rightarrow & H_1(H, B) \rightarrow H_1(G, B) \rightarrow H_1(G/H, B_H) \rightarrow 0,
\end{array}
$$

where $K_2(A) = \text{Coker}(\iota_\ast: H_2(H, A) \rightarrow H_2(G, A))$, $\iota: H \rightarrow G$ being the inclusion.

To make clear what the horizontal maps are, and to prove the rows exact, we make use of the homology spectral sequence

$$H_p(G/H, H_q(H, A)) \Rightarrow H_{p+q}(G, A).$$

The latter is completely dual to the usual Hochschild-Serre spectral sequence, and the edge homomorphisms $H_p(G, A) \rightarrow H_p(G/H, A_H)$

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and $H_q(H, A) \to H_q(G, A)$ are induced respectively by the obvious arrows $G \to G/H$ and $i: H \to G$. The exactness, then, at the last four places is just dual to the exactness of the so-called fundamental exact sequence in the cohomology of groups. The exactness at the second place and the definition of arrow $(\ast)$ are derived from a slightly subtler analysis of the spectral sequence. (This remark—whose analogue holds for cohomology—was first pointed out to me by G. P. Hochschild.) Simply, if

$$0 \subset F_0 \subset F_1 \subset F_2 = H_2(G, A)$$

is the filtration associated with the spectral sequence, then $F_2/F_0 = H_2(G, A)/\text{Im} \{ i_\ast: H_2(H, A) \to H_2(G, A) \} = K_2(A)$ and $F_1/F_0 = E_{2,1}^2$.

The latter, however, is a homomorphic image of $E_{2,1}^2 = H_1(G/H, H_1(H, A))$ since $d_{2,1}^1 = 0$.

By hypothesis the arrows (1), (4), (5) are isomorphisms. Moreover, there is the following commutative diagram with exact rows.

$$
\begin{array}{ccc}
H_2(H, A) & \to & H_2(G, A) \\
\downarrow (1) & & \downarrow (8) \\
\downarrow (7) & & \downarrow (2) \\
H_2(H, B) & \to & H_2(G, B) \\
\end{array}
$$

Since arrows (7), (8) are isomorphisms, so is (2). By two applications of the Five Lemma, arrows (3), (6) are isomorphisms. Thus since $G/H$ is cyclic, $H_n(G/H, f_\ast)$ is an isomorphism for all $n \geq 1$.

By the induction hypothesis we may assume that $H_n(U, f)$ is an isomorphism for all proper subgroups $U$ (in particular, for $H$), and for all $n \geq 1$. Hence it suffices to show that $H_n(G, f)$ is an isomorphism for all $n \geq 1$. To see this consider the morphism of homology spectral sequences induced by $f$. For the $E^2$ terms this gives arrows

$$H_p(G/H, H_q(H, A)) \to H_p(G/H, H_q(H, B))$$

which are isomorphisms for $(p, q) \neq (0, 0)$. This is true by the inductive hypothesis if $q > 0$, and it is what is proved above for $q = 0$. It now follows that the morphism of spectral sequences is an isomorphism, and the induced morphisms $H_n(G, f) (n > 0)$ at the end of the spectral sequence are isomorphisms. This completes the proof of the theorem.

**Remarks.** 1. The above theorem implies the theorem on cohomological triviality of modules. If $\hat{H}^\alpha(U, A)$ vanishes in two successive dimensions for all subgroups $U$, apply the above theorem to the zero morphism of $A$ onto 0. Since this and Tate's theorem are equivalent, we have yet another proof of Tate's theorem.
2. Let \( \hat{H}^n(U, A) \cong \hat{H}^n(U, B) \) in two successive dimensions and for all subgroups but do not assume the isomorphisms induced by a module homomorphism. It would not be reasonable to expect isomorphisms for all \( n \) and all subgroups. The following counterexample justifies our pessimism. Let \( G = G_p(a, b: a^3 = b^7 = 1, aba^{-1} = b^2) \); let \( A \) be \( \mathbb{Z} \) with trivial action and \( B \) the result of dimension shifting down two steps. Then \( \hat{H}^q(G, A; 7) = \hat{H}^{q-2}(G, B; 7) = 0 \) for \( q = 1, 2, 3, 4, 5 \) and \( \hat{H}^4(G, A; 7) = \hat{H}^4(G, B; 7) \neq 0 \).

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QUASI-INVERTIBLE PRIME IDEALS

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In this note \( R \) will denote a commutative ring with unit and a proper ideal of \( R \) is an ideal of \( R \) different from \((0)\) and \( R \). Nakano has shown that \( R \) is a Dedekind domain, provided that every proper prime ideal of \( R \) is invertible [1]. In [2], Krull defines a prime ideal \( P \) to be quasi-invertible provided \( PP^{-1} > P \), where \( > \) denotes proper containment and \( P^{-1} \) is the set of elements \( x \) in the total quotient ring of \( R \) such that \( xP \subset R \). The purpose of this note is to prove that Nakano's result remains valid when invertible is replaced by quasi-invertible. Examples are known of rank-two valuation rings in which the maximal ideal is invertible—hence, in Nakano's result, prime cannot be replaced by maximal.

**Lemma.** If \( P \) is an invertible prime ideal in \( R \) then \( \cap_n P^n \) is a prime ideal.

**Proof.** The proof is the same as that of the first part of Theorem 4 of [1].

**Theorem.** If every proper prime ideal of \( R \) is quasi-invertible, then \( R \) is a Dedekind domain.

**Proof.** If \( R \) is a field there is nothing to prove. Let \( M \) be an arbitrary proper maximal ideal of \( R \) and denote by \( R_M \) the quotient ring of \( R \) with respect to \( M \) (see [3, pp. 218–228]). Let \( N \) denote the ideal consisting of the elements \( x \in R \) such that there exists an element \( m \in M \) such that \( mx = 0 \). Let \( h \) be the natural homomorphism from

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