THE POLYNOMIAL OF A DIRECTED GRAPH

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1. Introduction. In a recent paper [1], the concept of the polynomial of an undirected graph was introduced, and it was pointed out that (i) a graph has a polynomial if and only if it is regular and connected, and (ii) various previous studies (see the references in [1]) were special cases of the problem: find all graphs having the same polynomial.

In this paper, we prove the analogue of (i) for directed graphs, and, in addition, obtain some results of type (ii) for a class of directed graphs arising from a mesh on a torus.

2. On the existence of polynomials. Let $G$ be a directed graph on $n$ vertices, with at most one edge from vertex $i$ to vertex $j$, and no edge from $i$ to $i$. For each vertex $i$, let $d_i$ be the number of edges with terminal vertex $i$, $e_i$ be the number of edges with initial vertex $i$. $G$ is said to be strongly regular if $d_i = e_i = d$, $i = 1, \ldots, n$; $G$ is said to be strongly connected if, for any vertices $i$ and $j$, $i \neq j$, there is a directed path from $i$ to $j$.

Let $A(G) = A$ be the adjacency matrix of $G$, i.e.,

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u$ be the vector of order $n$ every entry of which is unity, $J$ the matrix of order $n$ every column of which is $u$.

**Theorem 1.** (i) There exists a polynomial $P(x)$ such that

$$(2.1) \quad J = P(A)$$

if and only if $G$ is strongly connected and strongly regular.

(ii) The unique polynomial of least degree satisfying (2.1) is $nS(x)/S(d)$ where $(x - d)S(x)$ is the minimal polynomial of $A$ and $d$ is the valence of $G$.

(iii) If $P(x)$ is that polynomial of least degree satisfying (2.1), then

the valence of $G$ is the greatest real root of $P(x) = n$.

**Proof.** Assume (2.1). Let $i, j$ be distinct vertices of $G$. By (2.1), there is some integer $k$ such that $A^k$ has a positive entry in position
(i, j), i.e., there is some k-step path from i to j. So G is strongly connected. Further, from (2.1), J commutes with A. But the (i, j)th entry of AJ is $e_i$, and the (i, j)th entry of JA is $d_j$. Thus $e_i = d_j$ for all i and j, so G is strongly regular.

To prove the converse of (i), assume G strongly connected and strongly regular. From the strong regularity, u is a left and right eigenvector of A, corresponding to the eigenvalue d. Hence, if d has multiplicity greater than 1, it must have at least one more eigenvector associated with it. But from the strong connectedness, using a standard argument [1], u is the only eigenvector corresponding to d. It follows that, if $R(x)$ is the minimal polynomial of A, and if $S(x) = R(x)/(x - d)$ then $S(d) \neq 0$. We then have

$$0 = R(A) = (A - dI)S(A).$$

Since $R(A)v = 0$ for all vectors v, it follows from (2.2) that

$$(A - dI)S(A)v = 0,$$

so $S(A)v = \alpha u$ for some $\alpha$.

If $(v, u) = 0$ then $(A^kv, u) = (v, (A^t)^k u) = d^k(v, u) = 0$ for every k and so $(S(A)v, u) = 0$. Therefore, $0 = (S(A)v, u) = (\alpha u, u) = n\alpha$, i.e., $\alpha = 0$.

Thus $S(A)v = 0$ for all v such that $(v, u) = 0$; further, $S(A)u = S(d)u$. Hence $nS(A)/S(d) = J$, i.e. a polynomial which will accomplish (2.1) is

$$(2.3) P(x) = \frac{n}{S(d)} S(x).$$

This completes the proof of (i); (ii) follows since (2.3) has smaller degree than the minimal polynomial of A.

To prove (iii) we note that A is non-negative and has row and column sums d. Thus, by [2], the eigenvalues of A are of all of absolute value $\leq d$. The roots of $P(x)$ are eigenvalues of A and hence for real $x > d$, $|P(x)|$ is a monotone increasing function of x. From (2.3), $P(d) = n$ and so, since $P(x)$ is a real polynomial, $P(x) > n$ for $x > d$.

This completes the proof of the theorem. We call (2.3) the polynomial belonging to G (and also say that G belongs to the polynomial).

3. A graph on a torus. For any positive integer t let $G_t$ be the graph whose vertices are all ordered pairs $(i, j)$ of residues mod t and whose edges go from $(i, j)$ to $(i, j + 1)$ and $(i + 1, j)$ for all i, j. Clearly $G_t$ is strongly regular of valence 2, and strongly connected. We now derive its polynomial.
Let $X, \mu$ be arbitrary (not necessarily distinct) $t$th roots of unity. Let $v$ be the vector whose $(i, j)$th component is $X^i \mu^j$. If $A$ is the adjacency matrix of $G_t$ then $Av = (\lambda + \mu)v$. Further, different vectors $v_1, v_2$ have as their scalar product $\sum_{i,j} \lambda_i \mu_j \lambda_i \mu_j^2$, which is zero unless $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$. Hence the set of vectors corresponding to the $t^2$ choices of the pair $\lambda, \mu$ form a complete orthogonal set of right eigenvectors. From this it follows that $A$ is normal and that the minimal polynomial of $A$ has no repeated factors. Hence, by Theorem 1, the polynomial belonging to $G_t$ is

$$P_t(x) = \frac{t^2}{S_t(x)},$$

where

$$S_t(x) = \prod_{i} \frac{(x - \rho)}{x - 2},$$

the product being taken over all distinct $\rho$ of the form $\lambda + \mu$, where $\lambda, \mu$ are $t$th roots of unity. For example,

$$P_1(x) = \frac{1}{2} x(x + 2),$$

$$P_3(x) = \frac{1}{12} (x^3 + 1)(x^3 + 2x + 4),$$

$$P_4(x) = \frac{1}{80} x(x + 2)(x^2 + 4)(x^2 + 4).$$

4. Does $P_t(x)$ characterize $G_t$? In view of the investigations of comparable questions for undirected graphs, it is natural to ask: if $H$ is a graph with $t^2$ vertices, and $P_t(x)$ is the polynomial of $H$, is $H \cong G_t$? We know of no instance in which $H \not\cong G_t$, but have only been able to prove $H \cong G_t$ if $t$ is a prime or if $t = 4$. Before specializing to those cases, however, we begin with a few lemmas. We assume $H$ has $t^2$ vertices and belongs to $P_t(x)$, and $A$ is the adjacency matrix of $H$.

**Lemma 1.** $H$ is strongly connected and strongly regular of valence 2.

**Proof.** That $H$ is strongly regular and strongly connected follows from the fact that $H$ has a polynomial. By Theorem 1 (iii) the valence of $H$ and the valence of $G_t$ both equal the largest real root of $P(x) = n$ and so the valence of $H = n$ the valence of $G_t = 2$.

**Lemma 2.** The vertices of $H$ can be partitioned into $t$ sets $T_i$ ($i \in \mathbb{Z}_t$, \text{...})
the ring of residue classes mod \( t \), such that every edge in \( H \) goes from a vertex in \( T_i \) to a vertex in \( T_{i+1} \).

**Proof.** From the proof of Theorem 1, we know that 2 is an eigenvalue of \( A \) of multiplicity one, and every eigenvalue is of absolute value at most 2. Because 2\( \lambda \) is also an eigenvalue of \( A \) for \( \lambda \) any \( t \)th root of unity, it follows [2] that \( A \) can be conceived as having the appearance

\[
\begin{bmatrix}
0 & A_0 & 0 & 0 \\
0 & A_1 & 0 & 0 \\
0 & 0 & A_{t-2} & 0 \\
A_{t-1} & 0 & 0 & 0
\end{bmatrix}
\]

(4.1)

where each diagonal block of 0’s is square. But each \( A_i \) must also be square, since the numbers of 1’s in \( A_i \) is twice the number of rows of \( A_i \) and also twice the number of columns. Thus \( A_i \) is of order \( t \), which implies the lemma.

**Lemma 3.** Let \( t > 2 \), let \( \omega \) be a primitive \( t \)th root of unity and let \( \lambda \) be any \( t \)th root of unity. Then for any \( r, s \) with \( (s, t) = 1 \), \( 1 + \omega^r \) and \( \lambda(1 + \omega^s) \) have the same multiplicities as eigenvalues of \( A \).

**Proof.** Let \( x \) be an eigenvector of \( A \) corresponding to the eigenvalue \( \alpha \), and let \( x = (x_0, \ldots, x_{t-1}) \) denote the partitioning of the coordinates of \( x \) corresponding to (4.1). We have \( A_i x_{i+1} = \alpha x_i \). Thus \( A_i (\lambda x_{i+1}) = \lambda x_i \). Thus \( (x_0, \lambda x_1, \ldots, \lambda^{t-1} x_{t-1}) \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \alpha \). Since the minimal polynomial of \( A \) has no repeated factors, the multiplicity of an eigenvalue is just the dimension of the corresponding space of eigenvectors and so the multiplicities of \( \alpha \) and \( \lambda \alpha \) are the same. Finally the multiplicities of \( 1 + \omega^r \) and \( 1 + \omega^s \) are the same, since these are algebraic conjugates and the characteristic polynomial of \( A \) is rational. This concludes the proof of the lemma.

Note in particular that 2\( \lambda \) is a simple eigenvalue of \( A \).

**Lemma 4.** Let \( A \) be of the form (4.1) and of rank \( r \). Then for \( 0 \leq i \leq t-1, j \geq 0 \), the rank of \( A_i A_{i+1} A_{i+2} \cdots A_{i+j} \) is \( r/t \), where addition of suffixes is taken mod \( t \).

**Proof.** Let \( m_i \) be the rank of \( A_i \). Then \( \sum_i m_i = \) the rank of \( A = r \). Since the minimal polynomial of \( A \) has no repeated factors, \( A = S^{-1}DS \) for some nonsingular \( S \) and diagonal \( D \). Hence, the rank of \( A^i \) is
also $r$. Now $A^t$ consists of diagonal blocks $A_0 A_1 \cdots A_{i-1}, A_1 A_2 \cdots A_{i-1} A_0, \cdots, A_{i-1} A_0 \cdots A_{i-2}$. The rank of each block is at most $m = \min_i m_i$; hence, $\sum_i m_i = r = t m$ and so $m_i = r/t$ for all $i$. This proves the lemma for $j = 0$. The result for $j > 0$ follows from a similar consideration of $A^t$.

**Lemma 5.** Let $t > 2$. If $A$ is normal, then $H \cong G_t$.

**Proof.** Let $K_i$ be the undirected bipartite graph whose vertices are the vertices of $T_i$ and $T_{i+1}$, as defined in Lemma 2, and which has an undirected edge joining $x \in T_i$ and $y \in T_{i+1}$ if and only if an edge of $H$ joins $x$ to $y$.

We first show that $K_i$ is a cycle of length $2t$. Since every vertex of $K_i$ is of valence 2, $K_i$ is the union of $p_i$ cycles for some $p_i \geq 1$. The matrix $A A^T$ has 4 as an eigenvalue with multiplicity $\sum_i p_i$. But since $A$ is normal, and $2X$ (for $X$ any $i$th root of unity) is a simple eigenvalue of $A$, $A A^T$ has 4 as an eigenvalue with multiplicity $t$. Hence $p_i = 1$, $i = 0, \ldots, t-1$, which was to be proven.

Next, we show that trace $A^t = 2t^2$. Since each $K_i$ is a complete cycle the eigenvalues of $A A^T$ are the union of the eigenvalues of $t$ matrices of order $t$ of the form $2I + P_i + P_i^T$ ($i = 0, \ldots, t-1$), where each $P_i$ is a permutation matrix that represents a single cycle on $t$ letters. Therefore, $A A^T$ has: 4 as an eigenvalue with multiplicity $t$; $2 + \lambda + \bar{\lambda}$ as an eigenvalue with multiplicity $t$; for $\lambda = \exp(2\pi i k/t), k = 1, \ldots, [(t-1)/2]$; and if $t$ is even, 0 as an eigenvalue with multiplicity $t$.

Since $A$ is normal, these are the squares of the absolute values of the eigenvalues of $A$. Therefore, the number of eigenvalues of $A$ of a given absolute value (other than 2 or 0) is the same for each absolute value. We also know from Lemma 3 that all eigenvalues of the same absolute value occur equally often. It follows that $A$ has the same eigenvalues as the adjacency matrix for $G_t$. But the trace of the $t$th power of that matrix is $2t^2$, so trace $A^t = 2t^2$.

Since $K_0$ is a cycle of length $2t$, we may label the vertices of $T_0$ and $T_1$ as $(i, -i)$ and $(i+1, -i)$, respectively ($i \in \mathbb{Z}_t$), in such a way that the edges from $(i, -i)$ go to $(i+1, -i)$ and $(i, 1-i)$. Since $A$ is normal and there is just one vertex, namely $(i, -i)$, which is the initial vertex of edges to both $(i+1, -i)$ and $(i, 1-i)$, it follows that there is just one vertex, which we label $(i+1, 1-i)$, which is the terminal vertex of edges from both $(i+1, -i)$ and $(i, 1-i)$. We now have the vertices of $T_1$ and $T_2$ labelled in such a way that the edges from $(i+1, -i)$ go to $(i+2, -i)$ and $(i+1, 1-i)$. We may continue labelling in this fashion the vertices of $T_3, T_4, \ldots, T_{t-1}$. Let $p_{ij}$ be the number of paths of length $t-1$ from $(i, -i)$ to $(j, t-1-j)$.
Then \( p_{ij} \) is \( \binom{t-1}{m} \) where \( m \) is the least positive residue (mod \( t \)) of \( j-i \), for the normality of \( A \) implies that the count of paths mimics the Pascal triangle. If \( (\alpha_i, t-i-\alpha_i) \) and \( (\beta_i, t-1-\beta_i) \) are the vertices of \( T_{t-1} \) which are initial vertices of edges going to \( (i, -i) \), then the number of paths of length \( t \) from \( (i, -i) \) to itself is \( p_{i,\alpha_i} + p_{i,\beta_i} \). By hypothesis, trace \( A^t = 2t \). Since the diagonal blocks of \( A^t \) are cyclic permutations of the factors \( A_1, A_2, \ldots, A_t \), each block has the same trace, \( 2t \). Hence,

\[
\sum_{i=0}^{t-1} (p_{i,\alpha_i} + p_{i,\beta_i}) = 2t.
\]

Since each \( p_{i,\alpha_i} \) and \( p_{i,\beta_i} \) is at least 1, it follows that \( p_{i,\alpha_i} \) and \( p_{i,\beta_i} \) are exactly 1 and that \( \alpha_i, \beta_i \) are just \( i \) and \( i-1 \). We now have that the edges from \( (i, t-1-i) \) go to \( (i+1, -i-1) \) and \( (i, -i) \) and have completed an explicit isomorphism between \( G_t \) and \( H \).

**Theorem 2.** If \( t = 2, 4 \) or an odd prime, and \( H \) is a graph with \( t^2 \) vertices that belongs to \( P_t(x) \), then \( H \cong G_t \).

**Proof.** We shall continue to use the notations of the lemmas.

If \( t = 2 \) the classes \( T_i \) of Lemma 2 each have 2 elements; hence the only possible distribution of edges is that of \( G_2 \).

If \( t = 4 \) then the eigenvalues of \( A \) are \( \pm 2, \pm 2i, \pm 1 \pm i \) and 0. By Lemma 3 the eigenvalues \( \pm 2, \pm 2i \) are simple and the eigenvalues \( \pm 1 \pm i \) have the same multiplicity, \( m \) say. Since \( A \) is of order 16 the multiplicity of 0 is \( 12 - 4m \); hence \( m = 1 \) or 2. Suppose first that \( m = 1 \), i.e., the multiplicity of 0 is 8. Now, by Lemma 4, each \( A_i \) is of rank 2 and must therefore be of the form \( P_i B Q_i \), where \( P_i, Q_i \) are permutation matrices and

\[
B = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.
\]

Let \( J_4 \) be the \( 4 \times 4 \) matrix of all 1's. It may be readily verified that for a permutation matrix \( R \), \( BRB \) is one of \( 2B \), \( 2J_4 - 2B \) or \( J_4 \), and that \( J_4 R B \) is \( 2J_4 \). Hence

\[
A_1 A_2 A_3 A_4 = P_1 B Q_1 P_2 B Q_2 P_3 B Q_3 P_4 B Q_4 = 8P_1 B Q_4 \quad \text{or} \quad 8P_1 (J_4 - B) Q_4 \quad \text{or} \quad 4P_1 J_4 Q_4 = 4J_4.
\]

The third possibility cannot occur since, by Lemma 4, \( A_1 A_2 A_3 A_4 \) is of rank 2.
Now $A^4$ is of the form
\[
\begin{bmatrix}
A_1A_2A_3A_4 & 0 & 0 & 0 \\
0 & A_2A_3A_4A_1 & 0 & 0 \\
0 & 0 & A_3A_4A_1A_2 & 0 \\
0 & 0 & 0 & A_4A_1A_3A_2
\end{bmatrix}
\]
and, as in the proof of Lemma 5, each of the diagonal blocks has the same eigenvalues and hence the same trace. From (4.2), the elements of $A_1A_2A_3A_4$ are divisible by 8; similarly for $A_2A_3A_4A_1$, $A_3A_4A_1A_2$ and $A_4A_1A_3A_2$. It follows therefore that the trace of $A^4$ is a multiple of 32. On the other hand, the trace of $A^4 = \sum \lambda^4$, the sum being over the eigenvalues $\lambda$ of $A$. On the assumption that $m = 1$ these eigenvalues are, $2, 2i, -2, -2i, 1+i, 1-i, -1+i, -1-i$, and 0 with multiplicity 8. A direct computation shows that $tr(A^4) = 48$. This contradicts the conclusion that $32 \mid tr(A^4)$ and thus demonstrates the impossibility of the case $m = 1$.

In the remaining case for $t = 4$ the multiplicities of the eigenvalues are the same as those of the adjacency matrix of $G_t$. Hence the sum of the squares of the moduli of the eigenvalues of $A$ is $2^4$, which is the same as the sum of the squares of the elements of $A$. Therefore $A$ is normal. By Lemma 5, $H \cong G_t$.

Finally, if $t$ is an odd prime, the eigenvalues of $A$ are just $2\omega^r$ and $\omega^r + \omega^s$ for $0 \leq r < s < t$, and $\omega$ a primitive $t$th root of unity. We note that these numbers are all distinct. Now, by Lemma 3, $2\omega^r$ is a simple eigenvalue, and the eigenvalues $\omega^r + \omega^s$ all have the same multiplicity. This multiplicity must be 2 in order to account for all $t^2$ eigenvalues of $A$. We now have, as in the second case for $t = 4$, that $A$ is normal, and $H \cong G_t$.

References