EMBEDDING PUNCTURED MANIFOLDS

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Let $M$ be a compact differential manifold without boundary of dimension $m$.

**Theorem.** Let $M-x$ be embedded differentially in a differential manifold $N$ of dimension $n$, in such a way that the normal bundle of the embedding is fibre homotopically trivial. Then there is a map of degree one onto the smash product $N \to S^{n-m} \wedge M$. (The degree should be taken mod 2 if $M$ or $N$ is not orientable, and with respect to the compact cohomology of $N-Bd N$ if $N$ is not compact or has boundary.)

**Proof.** Let $D^m$ be a closed disk in $M$ such that $x \in D^m \subset M$. The normal $(n-m)$-dimensional disk bundle of $M-\text{Int} D^m$ in $M$ has as its total space a compact $n$-dimensional manifold $L$ with boundary. Let $K=L/Bd L$. $L$ can be regarded as a submanifold of $N$ by the tubular neighbourhood theorem. The map $N \to K$, which sends $N-L$ to a point and which sends $L \to K$ via the identification map, has degree one. We shall show that $K$ is homotopy equivalent to $S^{n-m} \wedge M$.

Let $L_1$ be the total space of the trivial $(n-m)$-dimensional disk bundle over $M-\text{Int} D^m$. Let $K_1=L_1/Bd L_1$. The fibre homotopy equivalence, referred to in the hypotheses, gives rise to a homotopy equivalence between $K$ and $K_1$. Now

$$K_1 = M \times D^{n-m}/D^m \times D^{n-m} \cup M \times S^{n-m-1}$$
$$= M \times S^{n-m}/D^m \times S^{n-m} \cup M \times *$$
$$= M \times S^{n-m}/* \times S^{n-m} \cup M \times *$$
$$= M \wedge S^{n-m}.$$ 

This proves the theorem.

**Corollary 1.** If $M=L(2n, q)$, the three-dimensional lens space, then $M-x$ cannot be differentially embedded in $S^4$.

**Proof.** If $M-x$ is embedded in $S^4$, then, by the theorem, there is a map of degree one

$$S^4 \to S^1 \wedge M.$$

But Puppe [1, p. 416], shows that this is not true. (Puppe shows that

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a certain cohomology operation from dimension two to dimension four in $S^1 \# M$ is nonzero.) This completes the proof.

Zeeman [2] has shown that if $M = L(2n+1, q)$, then $M - x$ can be differentially embedded in $S^4$. It is well known that no lens space can be embedded in $S^4$ and any orientable three-dimensional manifold can be embedded in $S^6$ [4]. The situation with regard to embeddings of lens spaces and punctured lens spaces is therefore completely solved.

**Corollary 2.** Let $M$ be a simply connected, stably parallelizable manifold. Let $M - x$ be differentially embedded in $S^n$, where $2n \geq 3(m+1)$ and let the normal bundle of the embedding be fibre homotopically trivial. Then $M$ can be differentially embedded in $S^n$ (and we can assume that the normal bundle of the embedding is fibre homotopically trivial).

**Proof.** By the theorem, we have a map of degree one

$$S^n \to S^{n-m} \wedge M.$$ 

Now $S^{n-m-1} \ast M$ is homotopy equivalent to $S^{n-m} \wedge M$ and there is a map of degree one of $S^{n-m-1} \ast M$ onto the Thom space of the trivial $(n-m)$-dimensional vector bundle over $M$. We therefore have a map of degree one of $S^n$ onto the Thom space of the trivial bundle over $M$.

We have therefore verified all the hypotheses of a theorem proved by Levine [3, Theorem 1], the conclusion of which is the same as the conclusion of our Corollary 2.

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**Bibliography**


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