A TOTALLY BOUNDED, COMPLETE UNIFORM SPACE IS COMPACT

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Let $X$ be a set and $U$ a uniformity on $X$. We will show that if
$(X, U)$ is totally bounded, every net in $X$ has a Cauchy subnet. For
each $d \in U$, let $S_0^d, \ldots, S_k^d$ be a finite covering of $X$ by $d$-spheres.
Let $T_d$ be the topology on $X$ having $S_0^d, \ldots, S_k^d$ as its subbasis.
Clearly the space $(X, T_d)$ is compact. Therefore, $Y = \prod_{d \in U} (X, T_d)$ is
compact.

Now, let $(\rho_i)$ be a net in $X$. Then $\Delta \circ (\rho_i)$ is a net in $Y$, where
$\Delta: X \to Y$ is the diagonal. By compactness, there exists a convergent
subnet, $(q_j)$, of $\Delta \circ (\rho_i)$. Then $\Delta^{-1} \circ (q_j)$ is a subnet of $(\rho_i)$ which
is clearly Cauchy.

Thus, if $(X, U)$ is also complete, every net in $X$ has a convergent
subnet, so $(X, U)$ is compact.

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Received by the editors March 12, 1964.