DIFFERENTIABLE ACTIONS OF COMPACT
ABELIAN LIE GROUPS ON $S^n$

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1. Introduction. In [9] P. A. Smith raises the following question: If a finite group $G$ acts effectively on the $n$-sphere $S^n$, must there also be some effective orthogonal action of $G$ on $S^n$? Stated another way, must all finite groups acting effectively on $S^n$ be isomorphic to subgroups of the orthogonal group $O(n+1)$? Smith has answered this question in the affirmative for the case where $G$ is an elementary $p$-group [8], [9]. The Corollary to Theorem 2 of this paper settles the case where $G$ is a compact abelian Lie group (in particular, a finite abelian group) and the action is assumed differentiable.

The proof of our main result is immediate if one assumes the existence of a fixed point, as evidenced by the following result which utilizes Bochner’s theorem on local linearity about a fixed point.

Theorem 1. Let $G$ be a compact Lie group operating effectively and differentiably on a differentiable $n$-manifold $X$. If there exits a point $x_0$ left fixed by every element of $G$, then $G$ is isomorphic to a subgroup of $O(n)$.

Proof. By Bochner’s theorem [5, p. 206], we may assume $G$ acts orthogonally (but not necessarily effectively) on some small closed $n$-cell $D$ with center $x_0$. $G$ leaves bdy $D = S^{n-1}$ invariant. If $G$ is not effective on $S^{n-1}$, then there must be a homeomorphism $g_0$ of finite order in $G$ which leaves $S^{n-1}$ pointwise fixed. Since $g_0$ acts linearly on $D$ and leaves $x_0$ fixed, it must also leave $D$ pointwise fixed. By Newman’s theorem [5, p. 223], $g_0$ must leave $X$ pointwise fixed, violating the effectiveness of $G$ on $X$. Hence $G$ acts orthogonally and effectively on $S^{n-1}$, and the theorem follows.

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2. Definitions and preliminaries. An action of a transformation group $G$ on a space $X$ is said to be effective if $gx = x$ for all $x$ in $X$ implies $g = e$, the identity element of $G$. All spaces considered will be compact Hausdorff spaces and the usual Čech cohomology will be used. Let $\mathbb{Z}_p$ denote the additive group of integers modulo a prime $p$. Our definition of a cohomology $n$-manifold over $\mathbb{Z}_p$ and of cohomology dimension over $\mathbb{Z}_p$ will be that given in [1]. An elementary $p$-group of rank $k$ is a group isomorphic to the direct sum of $k$ copies of $\mathbb{Z}_p$. Smith [7] has shown that if an elementary $p$-group $G$ acts effectively on a connected cohomology $n$-manifold $X$ over $\mathbb{Z}_p$, then each component of $F(G, X)$, the fixed-point set of $G$ on $X$, is a connected cohomology $r$-manifold over $\mathbb{Z}_p$ with $r < n$ for each $r$. In fact, if $p$ is an odd prime, each $n - r$ must be even. A generalized cohomology $n$-sphere over $\mathbb{Z}_p$ is a cohomology $n$-manifold over $\mathbb{Z}_p$ which has the global cohomology, coefficient group $\mathbb{Z}_p$, of $S^n$. Results of Smith [6], [7] show that if an elementary $p$-group $G$ acts effectively on a generalized cohomology $n$-sphere $X$ over $\mathbb{Z}_p$, then $F(G, X)$ is a generalized cohomology $r$-sphere over $\mathbb{Z}_p$, $r < n$. We shall need the following result from [3]:

**Lemma 1.** Let $G$ be an elementary $p$-group of rank $k$ acting effectively on a connected $n$-manifold $X$. Suppose $F(G, X)$ is nonempty and of cohomology dimension $r$. Then

$$k \leq \begin{cases} \frac{n - r}{2} & \text{for } p \neq 2, \\ n - r & \text{for } p = 2. \end{cases}$$

3. Main results.

**Theorem 2.** Let $G$ be isomorphic to the direct sum of $k$ copies of $\mathbb{Z}_2$ and $l$ copies of $\mathbb{Z}_{2p}$, $p$ an odd prime. If $G$ acts effectively and differentiably on $S^n$, then

$$k + 2l \leq n + 1.$$  

**Proof.** By the above-mentioned Smith result [8], [9], we may assume $k, l \geq 1$. We shall suppose $k + 2l = n + 2$ and arrive at a contradiction.

There exists a subgroup $H$ of $G$ which is an elementary 2-group of rank $k + l$. We consider the action of $H$ on $S^n$. As usual, the isotropy subgroup of $H$ at a point $x$ in $X$ is defined as the subgroup of $H$ consisting of all elements which leave $x$ fixed. By the results of [8] or [4], there must exist an isotropy subgroup $H_1$ of $H$ of rank $k + l - 1$. Since $H_1$ is an isotropy subgroup, $F(H, X)$ must be a proper subset
of \( F(H, X) \). Proceeding inductively, we may construct the following two decompositions

\[
\begin{align*}
(1) & \quad H = H_0 \supset H_1 \supset \cdots \supset H_{k+l-1} \supset H_{k+1} = e, \\
(2) & \quad \emptyset \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k+l-1} \subseteq F_{k+1} = S^n,
\end{align*}
\]

where

(i) \( H_i, 0 \leq i \leq k+l, \) is a subgroup of \( H \) of rank \( k+l-i \) which operates on \( S^n \) with fixed-point set \( F_i \).

(ii) Each \( F_i \) is a generalized cohomology sphere over \( \mathbb{Z}_2 \).

(iii) Each \( F_i \) is a compact manifold. This fact follows from the differentiability of the action by applying Bochner's theorem. Moreover, by (ii), each \( F_i \) of positive dimension is connected.

(iv) \( \dim F_i < \dim F_{i+1}, 0 \leq i \leq k+l-1. \)

We proceed to investigate the decomposition (2) of \( S^n \). From \( \emptyset \) to \( S^n \), there are exactly \( (n+2) - (k+l+1) = l-1 \) gaps in dimension, that is, dimensions not assumed by the \( F_i \)'s. It follows that \( \dim F_{i_0} \) is even (including 0) for some \( i_0 \), with \( \dim F_{i_0} \leq 2l-2 \); for, otherwise, there would have to be at least \( l \) gaps. Select then the first \( i_0 \) with \( \dim F_{i_0} \) even. As \( k \geq 1 \), we have \( \dim F_{i_0} \leq 2l-2 < n. \)

There exists a subgroup \( K \) of \( G \) which is an elementary \( p \)-group of rank \( l \). The action of \( K \) on \( S^n \) leaves \( F_{i_0} \) invariant. Since \( F_{i_0} \) is a generalized cohomology sphere over \( \mathbb{Z}_2 \), the Euler characteristic, \( \chi(F_{i_0}) \), of \( F_{i_0} \), which is independent of the coefficient field, must be equal to 2. Hence, \( \chi(F_{i_0}) \) is not congruent to 0 modulo \( p \), and by an easy extension of Floyd's result in [2] to actions of elementary \( p \)-groups, the action of \( K \) on \( F_{i_0} \) must have a fixed point. By Lemma 1, there exists a subgroup \( T \) of \( K \) such that each element of \( T \) leaves \( F_{i_0} \) pointwise fixed and such that \( K/T \) is effective on \( F_{i_0} \) with

\[
\Gamma(K/T) = \text{rank } (K/T) \leq \frac{\dim F_{i_0}}{2}.
\]

We shall say that \( T \) is completely noneffective on \( F_{i_0} \). Now

\[
\Gamma(T) \geq \Gamma(K) - \frac{\dim F_{i_0}}{2} = l - \frac{\dim F_{i_0}}{2} \geq l - \frac{2l-2}{2} = 1.
\]

We next construct a decomposition of \( T \),

\[
\begin{align*}
(3) & \quad T = T_{i_0} \supsetneq T_{i_0+1} \supsetneq \cdots \supsetneq T_{k+1} \supsetneq e,
\end{align*}
\]

such that \( T_j, i_0 \leq j \leq k+l, \) is completely noneffective on \( F_j \). As \( T_{k+1} \) is a nontrivial subgroup of \( G \) which is noneffective on \( F_{k+1} = S^n \), we obtain a contradiction to the effectiveness of \( G \).
We proceed with the construction of (3). Now at least \((\dim F_i) / 2\) gaps have been used in arriving to \(F_{i_0}\) in (2). Therefore there exist at most

\[
(l - 1) - \frac{\dim F_{i_0}}{2}
\]
gaps from \(F_i\) to \(F_{k+1}\) in (2). Let \(N = \dim F_{i_0} - \dim F_i\) and consider the following two cases.

(i) \(N = 1\). Now \(T\) leaves the compact connected manifold \(F_{i_0+1}\) invariant with fixed-point set containing \(F_i\). Since \(N = 1\) and each element of \(T\) is of prime order \(p\), \(p\) odd, we have that \(T\) is completely noneffective on \(F_{i_0+1}\) due to above-mentioned parity restrictions. In this case, choose \(T_{i_0+1} = T\).

(ii) \(N \geq 2\). In this case, there are \(N - 1\) gaps from \(F_i\) to \(F_{i_0+1}\) in (2). Again, \(T\) leaves \(F_{i_0+1}\) invariant with fixed-point set \(F_i\) containing \(F_i\). By Lemma 1, there exists a subgroup \(T_{i_0+1}\) of \(T\) which is completely noneffective on \(F_{i_0+1}\) with

\[
\Gamma(T/T_{i_0+1}) \leq \frac{\dim F_{i_0+1} - \dim F_i}{2}
\]

\[
\leq \frac{\dim F_{i_0+1} - \dim F_{i_0}}{2} = \frac{N}{2}.
\]

Consequently,

\[
\Gamma(T_{i_0+1}) \geq \Gamma(T) - \frac{N}{2} \geq \Gamma(T) - (N - 1).
\]

We see that each gap results in reducing the rank of \(T\) to that of \(T_{i_0+1}\) by at most one.

Proceeding inductively, and recalling that there are at most

\[
(l - 1) - \frac{\dim F_i}{2}
\]
gaps from \(F_i\) to \(F_{k+1}\), we obtain

\[
\Gamma(T_{k+1}) \geq \Gamma(T) - \left[ (l - 1) - \frac{\dim F_i}{2} \right]
\]

\[
\geq \left[ l - \frac{\dim F_i}{2} \right] - \left[ (l - 1) - \frac{\dim F_i}{2} \right] = 1.
\]

**Corollary.** Let \(G\) be a compact abelian Lie group acting effectively
and differentiably on $S^n$. Then $G$ is isomorphic to a subgroup of $O(n+1)$.

Proof. $G$ is the direct sum of a $q$-torus $T^q$ and a finite abelian group $R$. Choose a minimal set of generators of $R$, let $g(R)$ denote the total number of these generators and $h(R)$ denote the number of these generators of order 2. Suppose, first, that $h(R) = 0$. Then there exists for some odd prime $p$ a subgroup $G'$ of $G$ with $G'$ an elementary $p$-group of rank $q + g(R)$. Then, by Smith [8], [9],

$$q + g(R) \leq \left\lfloor \frac{n + 1}{2} \right\rfloor$$

and the Corollary follows. Suppose then that $h(R) \geq 1$. Then there exists a subgroup $G''$ of $G$ with $G''$ isomorphic to the direct sum of $h(R)$ copies of $Z_2$ and $[q + g(R) - h(R)]$ copies of $Z_{2p}$, for some odd prime $p$. The Corollary now follows from Theorem 2.

References


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