1. Introduction. Let $f$ be an $L^1$ function whose Fourier transform $\hat{f}$ is free of (real) zeros. We will refer to such a function $f$ as a Wiener kernel, and write $f \in W$. Let $s$ be an $L^\infty$ function which is slowly oscillating:

$$s(x) - s(y) \to 0 \quad \text{as } x \to \infty \text{ and } x - y \to 0.$$ 

Finally suppose that

$$(f * s)(x) = \int_{-\infty}^{\infty} f(x - t)s(t) \, dt \to 0 \quad \text{as } x \to \infty.$$ 

Then

$$s(x) \to 0 \quad \text{as } x \to \infty.$$ 

This is Pitt's form [4], [5] of Wiener's Tauberian theorem [7], [8].

The above Tauberian theorem is easily derived from a closure theorem, also due to Wiener (loc. cit.), which asserts that for any $f \in W$ the finite linear combinations of translates $f(x + \lambda)$, $\lambda$ real, are dense in $L^1$. Thus by the continuous linear functionals test, Wiener's Tauberian theorem is a consequence of the following

**Theorem A.** For any Wiener kernel $f$, the equation

$$f * g = 0, \quad g \in L^\infty,$$ 

implies that $g = 0$.

It is also possible, as indicated by Beurling [1], to prove directly that Wiener's theorem is a consequence of Theorem A.

A heuristic proof of Theorem A goes as follows. By Fourier transformation, equation (1) becomes $\hat{f}\hat{g} = 0$. Thus since $\hat{f}$ is free of zeros one must have $\hat{g} = 0$, and hence $g = 0$. The only difficulty with this approach is that for arbitrary $g \in L^\infty$, the Fourier transform $\hat{g}$ is a (tempered) distribution, and the product $\hat{f}\hat{g}$ is not defined in the usual theory (cf. [3], however). In the present note we indicate how one can get around this problem by replacing $f$ with a suitable testing function of rapid descent, that is, a function belonging to Schwartz's

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Received by the editors December 27, 1963.

1 Work supported by NSF grant G-10093 at the University of Wisconsin.

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space $S$ [6]. (It should be mentioned that Beurling has given several proofs of Theorem A [1], [2], and that the second one cited also employs a generalized Fourier transform of $g$.)

2. **Two simple lemmas.** Let $\phi$ be any testing function of rapid descent, and define functions $\phi_n$ by setting

$$
\phi_n(x) = \frac{1}{n} \phi\left(\frac{x}{n}\right), \quad n = 1, 2, \ldots
$$

**Lemma 1.** For $f \in L^1$ and $\phi_n$ as above,

$$
\|f * \phi_n - \tilde{f}(0) \phi_n\| \to 0 \quad \text{as } n \to \infty.
$$

**Proof.** The norm in question is given by

$$
\int_{-\infty}^{\infty} \left| f(t) \right| \left\{ \phi_n(x - t) - \phi_n(x) \right\} \, dt \leq \int_{-\infty}^{\infty} \left| f(t) \right| \rho_n(t) \, dt,
$$

where

$$
\rho_n(t) = \|\phi_n(x - t) - \phi_n(x)\| = \|\phi(y - t/n) - \phi(y)\|.
$$

It is clear that $\rho_n(t) \to 0$ for every fixed $t$, while $\rho_n(t) \leq 2\|\phi\|$. The lemma thus follows from Lebesgue's dominated convergence theorem.

**Lemma 2.** Suppose that $u$ and $v$ belong to $L^1$ and that $\|v\| < 1$. Then $\hat{u}/(1 + \theta)$ is the Fourier transform of an $L^1$ function $w$.

**Proof.** Consider the series

$$
u = u * v + u * v * v - \ldots
$$

Since

$$
\|u * v * \| \leq \|u\| \|v\|,
$$

the sum of the norms of the terms in (3) is finite. It follows that the series converges in $L^1$ to a function $w$ which has the desired Fourier transform.

3. **Proof of Theorem A.** Suppose that $f \in W$, and that $g \in L^\infty$ satisfies equation (1). We introduce a testing function $\phi$ of rapid descent whose Fourier transform $\hat{\phi}$ is equal to 1 on $[-1, 1]$ and equal to 0 outside $(-2, 2)$.

Defining $\phi_n$ by equation (2), Lemma 1 shows that we can choose an index $p$ so large that

$$
\|f * \phi_p - \tilde{f}(0) \phi_p\| < |\tilde{f}(0)|.
$$
It is clear that $\phi_p(x) = \phi(px) = 1$ for $|x| \leq 1/p$; we also note that $\phi_{2p}(x) = 0$ for $|x| \geq 1/p$.

We now set

$$u = \frac{1}{f(0)} \phi_{2p}, \quad v = \frac{1}{f(0)} \{f * \phi_p - f(0) \phi_p\}.$$

By Lemma 2 the quotient

$$\frac{u}{1 + v} = \frac{\phi_{2p}}{f(0) + f \phi_p - f(0) \phi_p} = \hat{\phi}_{2p}$$

is the Fourier transform of an $L^1$ function $w$. For this $w$ we will have $w * f = \phi_{2p}$, hence, by equation (1),

$$\phi_{2p} * g = w * f * g = 0. \tag{4}$$

Since $\phi_{2p}$ is a testing function of rapid descent we can take Fourier transforms in (4) to obtain

$$\hat{\phi}_{2p} \hat{g} = 0. \tag{5}$$

Observing that the testing function $\hat{\phi}_{2p}$ is equal to 1 for $|x| \leq 1/2p$, one derives from (5) that the distribution $\hat{g}$ is equal to 0 at least on the open interval $|x| < 1/2p$.

So far we have only used the nonvanishing of $f(0)$. However, equation (1) shows that the convolution of $f(x)e^{-ix\alpha}$ and $g(x)e^{-ix\beta}$ is equal to 0 for every real number $\alpha$, hence, by the preceding argument, the nonvanishing of $f(0)$ implies that $g$ vanishes in a neighborhood of the arbitrary point $\alpha$. We conclude that $g = 0$ on $(-\infty, \infty)$ and, hence, $g = 0$.

References


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