Proof. If $M$ is Ricci flat it is trivially Einstein and, since $L^2 - HL = 0$, one of the principal curvatures is zero.
If $R^* = bI$, then $L^2 - HL + bI = 0$. If $K = 0$, then there is a zero principal curvature and a unit principal vector $X$ with $LX = 0$. Hence $bX = 0$ so $R^* = 0$.

In the case $n = 3$, the characteristic polynomial $L^3 - HL^2 + JL - KI = 0$ implies $JL = 0$, and since $Lm = 0$ implies $J(m) = 0$, we have $J = 0$.

Bibliography


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ON PSEUDOMETRICS FOR GENERALIZED UNIFORM STRUCTURES

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In [1] Alfsen and Njåstad generalized the concept of a uniform structure $\mathcal{U}$ on a set $S$, replacing the intersection axiom for uniform structures by the weaker condition:

(0) Given subsets $A_1, \cdots, A_n$ of $S$ and $U_1, \cdots, U_n$ in $\mathcal{U}$, there exists $U$ in $\mathcal{U}$ such that $U(A_i) \subseteq U_i(A_i)$ for $i = 1, \cdots, n$. Our object is to characterize these structures in terms of pseudometrics.

Define a (generalized) gage on $S$ to be a nonvoid family $\mathcal{G}$ of pseudometrics on $S \times S$ such that

(1) Every pseudometric uniformly continuous with respect to $\mathcal{G}$ belongs to $\mathcal{G}$.

(2) If $\alpha$ and $\beta$ belong to $\mathcal{G}$ and both $\alpha$ and $\beta$ are totally bounded, then $\alpha \vee \beta$ belongs to $\mathcal{G}$.

Note that if we delete the total boundedness condition in (2), then $\mathcal{G}$ is just a gage for a proper uniform structure [2], [3]. For $\beta$ a pseudometric on $S \times S$, define $W_\beta = \beta^{-1}[0, 1)$.

Theorem. Given a gage $\mathcal{G}$ on $S$, define the class $\mathcal{U}$ of subsets $U$ of $S \times S$ by the condition

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(i) $U \subseteq \mathcal{U}$ iff $U \supseteq W_\beta$ for some $\beta$ in $\mathcal{G}$.

Then $\mathcal{U}$ is a generalized uniform structure for which

(ii) $\alpha \in \mathcal{G}$ iff $W_{n\alpha} \subseteq \mathcal{U}$ for every positive integer $n$.

Conversely, given a generalized uniform structure $\mathcal{U}$ on $S$, define the family $\mathcal{G}$ of pseudometrics $\alpha$ by (ii). Then $\mathcal{G}$ is a gage for which (i) holds.

**Lemma.** Given a pseudometric $\alpha$ on $S \times S$ and nonempty subsets $A$ and $B$ of $S$ such that $\alpha(A, B) \geq 1$, there exists a totally bounded pseudometric $\beta$ such that $\beta \leq \alpha$ and $\beta(A, B) = 1$.

To prove the lemma, let $f(s) = \alpha(s, A)[\alpha(s, A) + \alpha(s, B)]^{-1}$ on $S$ and define $\beta(x, y) = |f(x) - f(y)|$. A few simple computations show that $\beta$ has the desired properties.

To prove the theorem, let $\mathcal{G}$ satisfy (1) and (2) and define $\mathcal{U}$ by (i). That $\mathcal{U}$ has all the properties of a uniform structure except for the intersection axiom follows exactly as in the case of proper uniform structures. To prove (0) we may, in view of (i), assume that $U_i = W_\alpha$ for some $\alpha_i = \alpha$ in $\mathcal{G}$. Let $B_i = S - U_i(A_i)$. Since the conclusion of (0) will be trivial wherever $A_i$ or $B_i$ is empty, we may assume both are nonempty. Apply the lemma to get $\beta_i$ totally bounded with $\beta_i \leq \alpha_i$ and $\beta_i(A_i, B_i) = 1$. $\beta_i$ is in $\mathcal{G}$ by (1). Let $U = W_\beta$. Given $y$ in $U(A_i)$, $(x, y)$ is in $U$ for some $x$ in $A_i$. That is, $\beta_i(x, y) \leq \beta(x, y) < 1$ for some $x$ in $A_i$. So $\beta_i(y, A_i) < 1$. Since $\beta_i(A_i, B_i) = 1$, $y$ is not in $B_i$. That is, $y$ is in $U_i(A_i)$. Thus (0) holds and $\mathcal{U}$ is a generalized uniform structure.

To prove (ii), consider any $G$ in $\mathcal{G}$. Then $nG$ is in $\mathcal{G}$ by (1) and hence $W_{n\alpha}$ is in $\mathcal{U}$ by (i). Conversely, let $W_{n\alpha}$ belong to $\mathcal{U}$ for all $n$. By (i) there exists for each $n$ some $\beta$ in $\mathcal{G}$ such that $W_\beta \subseteq W_{n\alpha}$. Thus (1) implies $\alpha$ is in $\mathcal{G}$.

Given a generalized uniform structure $\mathcal{U}$, define $\mathcal{G}$ by (ii). We must prove (1), (2), and (i). For $\beta$ uniformly continuous relative to $\mathcal{G}$ and $m$ any positive integer, there exist $\alpha$ in $\mathcal{G}$ and a positive integer $n$ such that $W_{n\alpha} \subseteq W_{m\beta}$. Since $W_{n\alpha}$ belongs to $\mathcal{U}$ by (ii), so does $W_{m\beta}$. So $\beta$ is in $\mathcal{G}$ by (ii). Hence (1) holds just as in the case of proper uniform structures.

To prove (2) let $\alpha$ and $\beta$ be totally bounded members of $\mathcal{G}$. Let $\gamma = \alpha \vee \beta$. Since $\gamma$ is totally bounded, we can get a finite covering $S_1 \cup \cdots \cup S_k = S$ with diameters $\gamma[S_i] < 1/4$. Applying (0) to the sequences

$$
\left\{ S_1, \ldots, S_k, S_1, \ldots, S_k \right\}
\left\{ W_{2\alpha}, \ldots, W_{2\alpha}, W_{2\beta}, \ldots, W_{2\beta} \right\}
$$

we get $U$ in $\mathcal{U}$ such that...
Consider any \((x, y)\) in \(U\). Since \(x\) is in some \(S_i\), \(y\) is in the corresponding \(U(S_i)\). Hence (3) implies \(\gamma(y, S_i) < 3/4\). So \(\gamma(x, y) \leq \gamma(x, S_i) + \gamma(y, S_i) < 0 + 1/4 + 3/4 = 1\). That is, \(U \subseteq W_\gamma\). So \(W_\gamma\) belongs to \(\mathcal{U}\) whenever \(\alpha\) and \(\beta\) are totally bounded members of \(\mathcal{G}\). Using (1) we can apply this result to \(n\alpha\) and \(n\beta\) to conclude that \(W_n\) belongs to \(\mathcal{U}\). That is, \(\gamma\) is in \(\mathcal{G}\). So (2) holds.

To prove (i) let \(U\) be any member of \(\mathcal{U}\). Choose a sequence \(\{U_n\}\) in \(\mathcal{U}\) such that \(U_n = U_{n-1}\) and \(U_{n+1} \subseteq U_n \subseteq U\) for all \(n\). By the Metrization Lemma [3] there exists a pseudometric \(\beta\) such that

\[
U_{n+1} \subseteq W_\beta \subseteq U_n \quad \text{for all } n.
\]

\(\beta\) is in \(\mathcal{G}\) by (4) and (ii). Setting \(n = 1\) in (4) yields \(W_\beta \subseteq U\) which proves the direct implication in (i). The converse follows from (ii) since \(W_\beta\) is in \(\mathcal{U}\) if \(\beta\) is in \(\mathcal{G}\).

Using the lemma and [4], we obtain the following corollaries.

**Corollary 1.** For a given proximity relation, let \(\mathcal{G}\) be the associated precompact gage and \(\mathcal{G}\) be the associated total [1] gage. Then \(\mathcal{G}\) consists of all pseudometrics \(\alpha\) on \(S \times S\) such that every totally bounded pseudometric \(\beta\) satisfying \(\beta \leq \alpha\) belongs to \(\mathcal{G}\).

**Corollary 2.** A gage \(\mathcal{G}\) is total iff \(\mathcal{G}\) contains every pseudometric \(\alpha\) for which every totally bounded pseudometric \(\beta\) satisfying \(\beta \leq \alpha\) belongs to \(\mathcal{G}\).

**References**


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