PROOF. If $M$ is Ricci flat it is trivially Einstein and, since $L^a - HL = 0$, one of the principal curvatures is zero.

If $R^* = bI$, then $L^2 - HL + bI = 0$. If $K = 0$, then there is a zero principal curvature and a unit principal vector $X$ with $LX = 0$. Hence $bX = 0$ so $R^* = 0$.

In the case $n = 3$, the characteristic polynomial $L^a - HL^2 + JL - KI = 0$ implies $JL = 0$, and since $L_m = 0$ implies $J(m) = 0$, we have $J = 0$.

Bibliography


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ON PSEUDOMETRICS FOR GENERALIZED UNIFORM STRUCTURES

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In [1] Alfsen and Njåstad generalized the concept of a uniform structure $\mathcal{U}$ on a set $S$, replacing the intersection axiom for uniform structures by the weaker condition:

(0) Given subsets $A_1, \ldots, A_n$ of $S$ and $U_1, \ldots, U_n$ in $\mathcal{U}$, there exists $U$ in $\mathcal{U}$ such that $U(A_i) \subseteq U_i(A_i)$ for $i = 1, \ldots, n$. Our object is to characterize these structures in terms of pseudometrics.

Define a (generalized) gage on $S$ to be a nonvoid family $\mathcal{G}$ of pseudometrics on $S \times S$ such that

(1) Every pseudometric uniformly continuous with respect to $\mathcal{G}$ belongs to $\mathcal{G}$.

(2) If $\alpha$ and $\beta$ belong to $\mathcal{G}$ and both $\alpha$ and $\beta$ are totally bounded, then $\alpha \vee \beta$ belongs to $\mathcal{G}$.

Note that if we delete the total boundedness condition in (2), then $\mathcal{G}$ is just a gage for a proper uniform structure [2], [3]. For $\beta$ a pseudometric on $S \times S$, define $W_\beta = \beta^{-1}[0, 1)$.

Theorem. Given a gage $\mathcal{G}$ on $S$, define the class $\mathcal{U}$ of subsets $U$ of $S \times S$ by the condition

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(i) \( U \subseteq \mathcal{U} \) iff \( U \supseteq W_\beta \) for some \( \beta \) in \( \mathcal{G} \).

Then \( \mathcal{U} \) is a generalized uniform structure for which

(ii) \( \alpha \leq \mathcal{G} \) iff \( W_{n\alpha} \subseteq \mathcal{U} \) for every positive integer \( n \).

Conversely, given a generalized uniform structure \( \mathcal{U} \) on \( S \), define the family \( \mathcal{G} \) of pseudometrics \( \alpha \) by (ii). Then \( \mathcal{G} \) is a gage for which (i) holds.

**Lemma.** Given a pseudometric \( \alpha \) on \( S \times S \) and nonempty subsets \( A \) and \( B \) of \( S \) such that \( \alpha(A, B) \geq 1 \), there exists a totally bounded pseudometric \( \beta \) such that \( \beta \leq \alpha \) and \( \beta(A, B) = 1 \).

To prove the lemma, let \( f(s) = \alpha(s, A) \left[ \alpha(s, A) + \alpha(s, B) \right]^{-1} \) on \( S \) and define \( \beta(x, y) = |f(x) - f(y)| \). A few simple computations show that \( \beta \) has the desired properties.

To prove the theorem, let \( \mathcal{G} \) satisfy (1) and (2) and define \( \mathcal{U} \) by (i). That \( \mathcal{U} \) has all the properties of a uniform structure except for the intersection axiom follows exactly as in the case of proper uniform structures. To prove (0) we may, in view of (i), assume that \( U_i = W_\alpha \) for some \( \alpha \leq \alpha \) in \( \mathcal{G} \). Let \( B_i = S - U_i(A_i) \). Since the conclusion of (0) will be trivial wherever \( A_i \) or \( B_i \) is empty, we may assume both are nonempty. Apply the lemma to get \( \beta_i \) totally bounded with \( \beta_i \leq \alpha_i \) and \( \beta_i(A_i, B_i) = 1 \). \( \beta_i \) is in \( \mathcal{G} \) by (1). Let \( U = W_\beta \). Given \( y \in U(A) \), \( (x, y) \) is in \( U \) for some \( x \) in \( A \). That is, \( \beta(x, y) \leq \beta(x, y) < 1 \) for some \( x \) in \( A \). So \( \beta_i(x, y) < 1 \). Since \( \beta_i(A_i, B_i) = 1 \), \( y \) is not in \( B_i \). That is, \( y \) is in \( U_i(A_i) \). Thus (0) holds and \( \mathcal{U} \) is a generalized uniform structure.

To prove (ii), consider any \( \alpha \) in \( \mathcal{G} \). Then \( n\alpha \) is in \( \mathcal{G} \) by (1) and hence \( W_{n\alpha} \) is in \( \mathcal{U} \) by (i). Conversely, let \( W_{n\alpha} \) belong to \( \mathcal{U} \) for all \( n \). By (i) there exists for each \( n \) some \( \beta \) in \( \mathcal{G} \) such that \( W_\beta \subseteq W_{n\alpha} \). Thus (1) implies \( \alpha \) is in \( \mathcal{G} \).

Given a generalized uniform structure \( \mathcal{U} \), define \( \mathcal{G} \) by (ii). We must prove (1), (2), and (i). For \( \beta \) uniformly continuous relative to \( \mathcal{G} \) and \( m \) any positive integer, there exist \( \alpha \) in \( \mathcal{G} \) and a positive integer \( n \) such that \( W_{n\alpha} \subseteq W_{m\beta} \). Since \( W_{n\alpha} \) belongs to \( \mathcal{U} \) by (ii), so does \( W_{m\beta} \). So \( \beta \) is in \( \mathcal{G} \) by (ii). Hence (1) holds just as in the case of proper uniform structures.

To prove (2) let \( \alpha \) and \( \beta \) be totally bounded members of \( \mathcal{G} \). Let \( \gamma = \alpha \lor \beta \). Since \( \gamma \) is totally bounded, we can get a finite covering \( S_1 \cup \cdots \cup S_k = S \) with diameters \( \gamma[S_i] < 1/4 \). Applying (0) to the sequences

\[
\left\{ S_1, \cdots, S_k, S_1, \cdots, S_k \right\}
\left\{ W_{2\alpha}, \cdots, W_{2\alpha}, W_{2\beta}, \cdots, W_{2\beta} \right\}
\]

we get \( U \) in \( \mathcal{U} \) such that
Consider any \((x, y)\) in \(U\). Since \(x\) is in some \(S_i\), \(y\) is in the corresponding \(U(S_i)\). Hence (3) implies \(\gamma(y, S_i) < 3/4\). So \(\gamma(x, y) \leq \gamma(x, S_i) + \gamma(S_i) + \gamma(y, S_i) < 0 + 1/4 + 3/4 = 1\). That is, \(U \subseteq W_\gamma\). So \(W_\gamma\) belongs to \(\mathcal{U}\) whenever \(\alpha\) and \(\beta\) are totally bounded members of \(\mathcal{G}\). Using (1) we can apply this result to \(n\alpha\) and \(n\beta\) to conclude that \(W_n\) belongs to \(\mathcal{U}\). That is, \(\gamma\) is in \(\mathcal{G}\). So (2) holds.

To prove (i) let \(U\) be any member of \(\mathcal{U}\). Choose a sequence \(\{U_n\}\) in \(\mathcal{U}\) such that \(U_n = U_{n-1}\) and \(U_{n+1} \subseteq U_n \subseteq U\) for all \(n\). By the Metrization Lemma [3] there exists a pseudometric \(\beta\) such that

\[
U_{n+1} \subseteq W_{2^{-n}\beta} \subseteq U_n \quad \text{for all } n.
\]

\(\beta\) is in \(\mathcal{G}\) by (4) and (ii). Setting \(n = 1\) in (4) yields \(W_\beta \subseteq U\) which proves the direct implication in (i). The converse follows from (ii) since \(W_\beta\) is in \(\mathcal{U}\) if \(\beta\) is in \(\mathcal{G}\).

Using the lemma and [4], we obtain the following corollaries.

**Corollary 1.** For a given proximity relation, let \(\mathcal{G}\) be the associated precompact gage and \(\mathcal{G}\) be the associated total gage. Then \(\mathcal{G}\) consists of all pseudometrics \(\alpha\) on \(S \times S\) such that every totally bounded pseudometric \(\beta\) satisfying \(\beta \leq \alpha\) belongs to \(\mathcal{G}\).

**Corollary 2.** A gage \(\mathcal{G}\) is total if \(\mathcal{G}\) contains every pseudometric \(\alpha\) for which every totally bounded pseudometric \(\beta\) satisfying \(\beta \leq \alpha\) belongs to \(\mathcal{G}\).

**References**


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