PARTIAL HOMOMORPHIC IMAGES OF BRANDT GROUPOIDS

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The main purpose of the present note is to show (Theorem 2) that any regular $\mathcal{D}$-class of any semigroup is a partial homomorphic image of a Brandt groupoid. It follows from this that a semigroup with zero is a partial homomorphic image of a Brandt semigroup if and only if it is regular and 0-bisimple.

In the first section, an alternative formulation is given of the determination by H.-J. Hoehnke [1] of all partial homomorphisms of a Brandt groupoid into an arbitrary semigroup. This is first done (Theorem 1) for any completely 0-simple semigroup. The result is a straightforward generalization of Theorem 3.14 of [2], in which all partial homomorphisms of one completely 0-simple semigroup into another are determined. The present terminology is that of [2]; Hoehnke omits the adjective "partial." Basic definitions given in [2] will not be repeated here; likewise, references to the fundamental work of Brandt, Rees, Green, and Munn can be found in [2].

1. Partial homomorphisms of a completely 0-simple semigroup.
Let $S$ and $S^*$ be semigroups with zero elements 0 and 0*, respectively. A mapping $\theta$ of $S$ into $S^*$ is called a partial homomorphism if (i) $\theta 0 = 0^*$, and (ii) $(ab)\theta = (a\theta)(b\theta)$ for every pair of elements $a, b$ of $S$ such that $ab \neq 0$. The restriction of $\theta$ to $S\setminus 0$ is then a partial homomorphism of the partial groupoid $S\setminus 0$ into $S^*$ as defined in [2, p. 93]. By agreeing to the trivial convention (i), there is no essential distinction between partial homomorphisms of $S$ into $S^*$ and of $S\setminus 0$ into $S^*$. Moreover, we need not require that $S^*$ have a zero element; if it does not, we adjoin a zero element 0* to it for the application of (i).

The author's interest in partial homomorphisms originated in the fact that they arise naturally in the theory of extensions of semigroups [2, §4.4].

A partial homomorphism $\theta: S \to S^*$ evidently preserves regularity [2, p. 26] and Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{D},$ and $\mathcal{C}$ [2, p. 47]. It follows that if $S$ is regular and 0-bisimple (i.e., $S\setminus 0$ is a $\mathcal{D}$-class of $S$ [2, p. 76]), then $(S\setminus 0)\theta$ is contained in a regular $\mathcal{D}$-class $D$ of $S^*$. This is the case, in particular, if $S$ is completely 0-simple [2, Theorem

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2.51, p. 79]. Since a Brandt semigroup $B^0$ is just a completely 0-simple inverse semigroup [2, Theorem 3.9, p. 102], we conclude, finally, that if $\theta$ is a partial homomorphism of a Brandt groupoid $B = B^0\setminus 0$ into a semigroup $S^\ast$, then $B\theta$ is contained in a regular $\mathcal{D}$-class $D$ of $S$. One might think that these successive particularizations would result in some restriction on $D$, particularly if $\theta$ is onto; the object of this note is to show that this is not the case (Theorem 2 below).

Let $D$ be a regular $\mathcal{D}$-class of $S^\ast$. Let

$$\{ R_{i^*}: i^* \in I^\ast \} \quad \text{and} \quad \{ L_{\lambda^*}: \lambda^* \in \Lambda^\ast \}$$

be the $\mathcal{R}$-classes and $\mathcal{L}$-classes, respectively, of $S^\ast$ contained in $D$. Then $H_{i^*\lambda^*} = R_{i^*} \cap L_{\lambda^*}$ are the $\mathcal{H}$-classes of $S^\ast$ contained in $D$. We know that at least one of these must contain an idempotent, and so be a maximal subgroup of $S^\ast$ [2, Theorem 2.16, p. 59]; choose one such and call it $H^\ast = H_{i^*\lambda^*}, 1^* \in \mathcal{I}^\ast \cap \mathcal{H}^\ast$. For each $i^* \in I^\ast$, pick $r_{i^*} \in H_{i^*\lambda^*}$, and for each $\lambda^* \in \Lambda^\ast$, pick $q_{i^*} \in H_{i^*\lambda^*}$. Then [2, Theorem 3.4, p. 92], every element of $D$ is uniquely representable in the form

$$(1) \quad r_{i^*} q_{\lambda^*} \quad (x \in H^\ast; i^* \in I^\ast, \lambda^* \in \Lambda^\ast).$$

We regard the triple $(x; i^*, \lambda^*)$ as coordinates of the element (1).

By the Rees Theorem [2, Theorem 3.5, p. 94], a completely 0-simple semigroup can be represented as a regular Rees $I \times \Lambda$ matrix semigroup $\mathcal{M}_0(G; I, \Lambda; P)$ over a group with zero $G^0$, and with $\Lambda \times I$ sandwich matrix $P = (p_{i^*\lambda^*})$. The elements of $\mathcal{M}_0$ can be represented as triples $(a; i, \lambda)$ multiplying according to

$$(2) \quad (a; i, \lambda)(b; j, \mu) = (a \phi_{i^*}(\omega_{i^*} b); i, \mu) \quad (a, b \in G^0; i, j \in I; \lambda, \mu \in \Lambda).$$

In fact, the proof of the Rees Theorem amounts to coordinatizing the $\mathcal{D}$-class $\mathcal{M}_0(0)$. It should be remarked that, for an arbitrary regular $\mathcal{D}$-class $D$, the elements (1) do not have a simple law of multiplication like (2).

**Theorem 1.** Let $S$ be a completely 0-simple semigroup represented as a regular Rees $I \times \Lambda$ matrix semigroup $\mathcal{M}_0(G; I, \Lambda; P)$. Let $\theta$ be a partial homomorphism of $S$ into a semigroup $S^\ast$. Then $(S\setminus 0)\theta$ is contained in a regular $\mathcal{D}$-class $D$ of $S$. Let $D$ be coordinatized as in (1). Then

$$(3) \quad (a; i, \lambda)\theta = r_{i^*\lambda^*}(\phi_{i^*}(\omega_{i^*} a); i, \lambda) \quad (a \in G; i \in I, \lambda \in \Lambda),$$

where

(i) $\phi: I \rightarrow I^\ast$ and $\psi: \Lambda \rightarrow \Lambda^\ast$ are mappings such that if $p_{i^*} \neq 0$ then $q_{\lambda^*} \in H^\ast$;

(ii) $\omega: G \rightarrow H^\ast$ is a (group) homomorphism;
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(iii) $u: I \to H^*$ and $v: \Lambda \to H^*$ are mappings such that if $\mu_i \neq 0$ then

$$\phi \circ \omega = v \circ (q \circ r \circ u).$$

The mappings $\phi, \psi, \omega, u, v$ are uniquely determined by $\theta$. Conversely, if mappings $\phi, \psi, \omega, u, v$ are given satisfying (i), (ii), and (iii), then (3) defines a partial homomorphism $\theta$ of $S\setminus 0$ into $D$.

Proof. The proof is so much like that of Theorem 3.14 of [2, p. 109], that we give only the outline. We can assume that the entry $\rho_{ii}$ of $P$ is not zero. The mappings $\phi$ and $\psi$ are determined by

$$R \subseteq R_{\phi}, \quad L \subseteq L_{\psi},$$

where \{ $R_i: i \in I$ \} are the $\alpha$-classes, and \{ $L_\lambda: \lambda \in \Lambda$ \} are the $\xi$-classes, of $S$. This implies that

$$(a; i, \lambda) \theta = r_i q_{i} \phi$$

for some $x$ in $H^*$. If $\rho_{ii} \neq 0$, then $(\rho_{ii}^{-1}; i, \lambda) \theta$ is an idempotent in $H_i q_i \phi$, and it follows that $q_i r_i \phi \in H^*$ [2, Theorem 2.17, p. 59]. Defining $\omega: G \to H^*$ by

$$(p_{1 \mu}^{-1}; a; 1, 1) \theta = r_{1 \phi} h_{0}^{-1} (a \omega) q_{1 \phi} \quad (h_0 = q_{1 \phi} r_{1 \phi}),$$

a brief calculation, using the uniqueness of the representation (1), shows that $\omega$ is a homomorphism. For each $i \in I$ and $\lambda \in \Lambda$ we define $u_i$ and $v_\lambda$ in $H^*$ by

$$(e; i, 1) \theta = r_i u_i q_{i \phi},$$

$$(\rho_{ii}^{-1}; 1, \lambda) \theta = r_{1 \phi} h_{0}^{-1} v_{\phi} q_{1 \phi}.$$

Applying $\theta$ to

$$(a; i, \lambda) = (e; i, 1) (\rho_{ii}^{-1}; a; 1, 1) (\rho_{ii}^{-1}; 1, \lambda),$$

we obtain (3). Applying $\theta$ to (2) and using (3), again with the uniqueness of (1), we obtain (4). This last step can be inverted to yield the converse part of the theorem.

From a constructive point of view, Theorem 1 has the drawback that, for given $\phi, \psi$, and $\omega$ satisfying (i) and (ii), there is no assurance that $u$ and $v$ can be found so as to satisfy (iii). This drawback disappears, however, when $S$ is a Brandt semigroup $B^0$. Here we can assume $B^0 = \Phi(G; I, I; \Delta)$, where $\Delta = (\delta_{ij})$ is the $I \times I$ identity matrix over $G^0$ [2, Theorem 3.9, p. 102]. The condition (4) now reduces to

$$e^* = v q_{i \phi} r_{i \phi} u_i \quad (all \ i \in I),$$

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where $e^*$ is the identity element of $H^*$; or, what is equivalent, to

$$v_i = u^{-1}(q_{i^*}(r_{i^*}))^{-1}. \tag{5}$$

We note that $q_{i^*}(r_{i^*}) \in H^*$ by (i). Thus we can always satisfy (iii) by choosing $u: I \to H^*$ arbitrarily, and then defining $v: I \to H^*$ by (5). Formula (3) becomes

$$v_i = u^{-1}(q_{i^*}(r_{i^*}))^{-1}q_{i^*}. \tag{6}$$

This differs from Hoehnke's formula (16) of [1, Part III, p. 97], chiefly because a definite coordinate system has been chosen for $D$, independent of $\theta$.

Now let $D$ itself be a Brandt groupoid, say

$$D = B^* = \{M_0(H^*; I^*, I^*; A^*)\}. \tag{7}$$

Let us use square brackets to represent the elements $[x^*; i^*, j^*]$ of $B^*$. It is natural to choose $r^* = [e^*; i^*, 1^*]$ and $q^* = [e^*; 1^*, i^*]$. We then have $q^*r^* = [e^*; 1^*, 1^*]$, while $q^*r^* = 0$ in $B^*$, or is undefined in $B^*$, if $i^* \neq j^*$. Hence condition (i) of Theorem 1 requires that

$$v_i = u^{-1}, \tag{5}$$

and (6) becomes

$$v_i = u^{-1}. \tag{7}$$

Thus every partial homomorphism $\theta$ of one Brandt groupoid, $B$, into another, $B^*$, is given by (7) in terms of (i) an arbitrary mapping $\phi: I \to I^*$, (ii) an arbitrary homomorphism $\omega: G \to H^*$, and (iii) an arbitrary mapping $u: I \to H^*$. (7) is equivalent to Hoehnke's formula (22) in [1, Part I, p. 164]. It can also be obtained by specialization from Theorem 3.14 of [2].

2. Partial homomorphic images of Brandt groupoids. We come now to the main result of the present note.

**THEOREM 2.** Any regular $\mathcal{D}$-class of any semigroup is a partial homomorphic image of some Brandt groupoid.

**PROOF.** Let $D$ be a regular $\mathcal{D}$-class of a semigroup $S$. Let

$$\{R_i: i \in I\} \quad \text{and} \quad \{L_\lambda: \lambda \in \Lambda\}$$

be the $\mathfrak{R}$-classes and $\mathcal{L}$-classes, respectively, of $S$ contained in $D$. As usual, we may assume that $I$ and $\Lambda$ have an element $1$ in common such that $H_{11} = R_1 \cap L_1$ is a group. But now we shall also assume, as we evidently may, that $I$ and $\Lambda$ are otherwise disjoint: $I \cap \Lambda = \{1\}$.

As usual, choose $r_i$ in $H_{1i}$ and $q_i$ in $H_{i1}$ in any way, for $i$ in $I \setminus 1$ and $\lambda$ in $\Lambda \setminus 1$, and choose $r_1 = q_1 = e_{11}$, the identity element of $H_{11}$. As
in (1), without the stars, every element of $D$ is uniquely representable in the form

$$r_i a q_{\lambda} \quad (a \in H_i; i \in I, \lambda \in \Lambda).$$

For $i$ in $I \setminus 1$ and $\lambda$ in $\Lambda \setminus 1$, let $q_i$ be any inverse of $r_i$ in $R_i$, and let $r_{\lambda}$ be any inverse of $q_{\lambda}$ in $L_{\lambda}$. Then

$$q_i r_{\lambda} = e_{11} \quad (a \in I \cup \Lambda).$$

Let $B = \mathfrak{B}(H_i; I \cup \Lambda, I \cup \Lambda; \Delta) \setminus 0$. Denote the elements of $B$ by triples $(a; \alpha, \beta)$. Multiplication in $B$ is given by

$$a; \alpha, \beta)(b; \beta, \gamma) = (ab; \alpha, \gamma) \quad (a, b \in H_i; \alpha, \beta, \gamma \in I \cup \Lambda).$$

Products $(a; \alpha, \beta)(b; \beta', \gamma)$ with $\beta \neq \beta'$ are not defined in $B$ (and are zero in $B^0$). Define $\theta: B \to D$ as follows:

$$(a; \alpha, \beta)\theta = r_{a} a q_{\beta} \quad (a \in H_i; \alpha, \beta \in I \cup \Lambda).$$

Then, because of (9),

$$(a; \alpha, \beta)\theta(b; \beta, \gamma)\theta = r_{a} a q_{\beta} r_{b} q_{\gamma} = r_{a b} q_{\gamma} \quad (ab; \alpha, \gamma)\theta.$$

From this and (10), it follows that $\theta$ is a partial homomorphism of $B$ into $D$. Moreover, $B \theta = D$, since $B \theta$ contains all the elements $r_{a} a q_{\lambda}$ of (8).

As described in §3.3 of [2], if we adjoin a zero element 0 to a Brandt groupoid $B$, defining $a b = 0$ if $a b$ is undefined in $B$, we obtain a Brandt semigroup $B^0$, that is, a completely 0-simple inverse semigroup. The following is immediate from Theorem 2 and the first assertion in Theorem 1.

**Corollary 1.** A semigroup with zero is a partial homomorphic image of some Brandt semigroup if and only if it is regular and 0-bisimple.

As defined in [2, p. 93], a partial isomorphism is a partial homomorphism which is one-to-one and onto. Not every regular $D$-class is a partial isomorphic image of some Brandt groupoid, and the question of telling which ones are remains unsettled. The next theorem gives a sufficient condition.

**Theorem 3.** Let $D$ be a regular $D$-class of a semigroup $S$ with the property that it is possible to set up a one-to-one correspondence between the $R$-classes $R$ of $D$ and the $\mathcal{L}$-classes $L$ of $D$ such that if $R$ and $L$ correspond, then $R \cap L$ contains an idempotent. Then $D$ is a partial isomorphic image of the Brandt groupoid having the same structure group as $D$ and the same number of $R$-classes (and $\mathcal{L}$-classes) as $D$. 
Proof. By hypothesis, we can index the \( R \)-classes and the \( \mathcal{L} \)-classes of \( D \) by the same index set \( I \), such that for each \( i \) in \( I \), \( R_i \cap L_i \) contains an idempotent \( e_i \). The \( \mathcal{C} \)-class \( H_{ij} = R_i \cap L_i \) is then the maximal subgroup \( H_{ij} \) of \( S \) containing \( e_{ij} \). Let \( 1 \in I \), and pick \( q_i \) in \( H_{ij} \) in any way for \( i \) in \( I \setminus 1 \), and let \( q_i = e_i \). Let \( q_i' \) be the inverse of \( q_i \) in \( H_{ij} \); such exists since both \( H_{ij} \) and \( H_{ji} \) contain idempotents [2, Theorem 2.18, p. 60]. Take \( B = \mathfrak{M}(H_{ij}; I, I; \Delta) \setminus 0 \) and define \( \theta: B \to D \) by

\[
(a; i, j) \theta = q_i' a q_i \quad (a \in H_{ij}; i, j \in I).
\]

Since every element of \( D \) is uniquely expressible in the form on the right-hand side of (11), and \( q_i q_i' = e_i \), we see at once that \( \theta \) is a partial isomorphism of \( B \) onto \( D \).

\( B \) is unique, to within isomorphism, since any Brandt groupoid is completely determined by its structure group and the cardinal number of its \( R \)-classes (or \( \mathcal{L} \)-classes).

Corollary 2. Every 0-bisimple inverse semigroup \( S \) is a partial isomorphic image of the Brandt semigroup having the same structure group as \( S \) and the same number of idempotents as \( S \).

Proof. The hypothesis of Theorem 3 is satisfied by any inverse semigroup [2, Corollary 2.19, p. 60]. For 0-bisimple inverse semigroups, in particular, for Brandt semigroups, the sets of \( R \)-classes, \( \mathcal{L} \)-classes, and nonzero idempotents all have the same cardinal.

We conclude with an example to show that a regular 0-bisimple semigroup may be a partial isomorphic image of a Brandt semigroup, but not of one having the same structure group.

Let \( B = \mathfrak{M}(E; I, I; \Delta) \setminus 0 \), where \( E = \{e\} \) is a one-element group, and \( I = \{1, 2\} \). Let \( S \setminus 0 = H \times E \), where \( H \) is a cyclic group \( \{e, a\} \) of order 2, and \( E \) is a right zero semigroup of order 2. We may represent the elements of \( S \) as pairs \((x; i)\) with \( x \in H, i \in I \), multiplying as follows:

\[
(x; i)(y; j) = (xy; j) \quad (x, y \in H; i, j \in I).
\]

Define \( \theta: B^0 \to S \) by

\[
(e; 1, 1) \theta = (e; 1), \quad (e; 1, 2) \theta = (a; 2)
\]

\[
(e; 2, 1) \theta = (a; 1), \quad (e; 2, 2) \theta = (e; 2)
\]

and \( 0 \theta = 0 \). Clearly \( \theta \) is one-to-one and onto, and it is easy to verify that it is a partial homomorphism.

On the other hand, \( S \) cannot be a partial isomorphic image of any Brandt semigroup \( B^0 \) having structure group of order 2. For \( B \) must then have order twice a square, whereas \( S \setminus 0 \) has order 4.
EVERY STANDARD CONSTRUCTION IS INDUCED
BY A PAIR OF ADJOINT FUNCTORS

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In this note, we prove the converse of the following result of
P. Huber [2]. Let $F: \mathcal{K} \to \mathcal{E}$ and $G: \mathcal{E} \to \mathcal{K}$ be covariant adjoint func-
tors, that is, functors such that there exist two (functor) morphisms
$\xi: I \to GF$ and $\eta: FG \to I$ satisfying the relations

1. $(\eta * F) \circ (F * \xi) = \iota * F,$
2. $(G * \eta) \circ (\xi * G) = \iota * G.$

Then, the triple $(C, k, p)$ given by

$$C = FG, \quad k = \eta \quad \text{and} \quad p = F * \xi * G$$

is a standard construction in $\mathcal{E}$, that is, $C$ is a covariant functor,
$k: C \to I$ and $p: C \to C^2$ are (functor) morphisms, and the following relations hold:

3. $(k * C) \circ p = (C * k) \circ p = \iota * C,$
4. $(p * C) \circ p = (C * p) \circ p.$

This standard construction is said to be induced by the pair of adjoint func-
tors $F$ and $G.$ For further explanation of the notation and terminology, see [2], or the appendix of [1].

Theorem. Let $(C, k, p)$ be a standard construction in a category $\mathcal{E}.$
Then there exists a category $\mathcal{K}$ and two covariant functors $F: \mathcal{K} \to \mathcal{E}$
and $G: \mathcal{E} \to \mathcal{K}$ such that

(i) $F$ is (left) adjoint to $G,$
(ii) $(C, k, p)$ is induced by $F$ and $G.$

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