PARTIAL HOMOMORPHIC IMAGES OF
BRANDT GROUPOIDS

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The main purpose of the present note is to show (Theorem 2) that any regular \( \mathcal{D} \)-class of any semigroup is a partial homomorphic image of a Brandt groupoid. It follows from this that a semigroup with zero is a partial homomorphic image of a Brandt semigroup if and only if it is regular and 0-bisimple.

In the first section, an alternative formulation is given of the determination by H.-J. Hoehnke \[1\] of all partial homomorphisms of a Brandt groupoid into an arbitrary semigroup. This is first done (Theorem 1) for any completely 0-simple semigroup. The result is a straightforward generalization of Theorem 3.14 of \[2\], in which all partial homomorphisms of one completely 0-simple semigroup into another are determined. The present terminology is that of \[2\]; Hoehnke omits the adjective "partial." Basic definitions given in \[2\] will not be repeated here; likewise, references to the fundamental work of Brandt, Rees, Green, and Munn can be found in \[2\].

1. Partial homomorphisms of a completely 0-simple semigroup.

Let \( S \) and \( S^\ast \) be semigroups with zero elements 0 and 0\(^\ast\), respectively. A mapping \( \theta : S \to S^\ast \) is called a partial homomorphism if (i) \( 0 \theta = 0^\ast \), and (ii) \( (ab) \theta = (a \theta)(b \theta) \) for every pair of elements \( a, b \) of \( S \) such that \( ab \neq 0 \). The restriction of \( \theta \) to \( S \setminus 0 \) is then a partial homomorphism of the partial groupoid \( S \setminus 0 \) into \( S^\ast \) as defined in \([2, p. 93]\). By agreeing to the trivial convention (i), there is no essential distinction between partial homomorphisms of \( S \) into \( S^\ast \) and of \( S \setminus 0 \) into \( S^\ast \). Moreover, we need not require that \( S^\ast \) have a zero element; if it does not, we adjoin a zero element 0\(^\ast\) to it for the application of (i).

The author's interest in partial homomorphisms originated in the fact that they arise naturally in the theory of extensions of semigroups \([2, \S 4.4]\).

A partial homomorphism \( \theta : S \to S^\ast \) evidently preserves regularity \([2, p. 26]\) and Green's relations \( \mathcal{R}, \mathcal{L}, \mathcal{D}, \) and \( \mathcal{C} \) \([2, p. 47]\). It follows that if \( S \) is regular and 0-bisimple (i.e., \( S \setminus 0 \) is a \( \mathcal{D} \)-class of \( S \) \([2, p. 76]\)), then \( (S \setminus 0) \theta \) is contained in a regular \( \mathcal{D} \)-class \( D \) of \( S^\ast \). This is the case, in particular, if \( S \) is completely 0-simple \([2, \text{Theorem}\]

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Since a Brandt semigroup $B^\circ$ is just a completely 0-simple inverse semigroup [2, Theorem 3.9, p. 102], we conclude, finally, that if $\theta$ is a partial homomorphism of a Brandt groupoid $B = B^\circ \setminus 0$ into a semigroup $S^*$, then $B\theta$ is contained in a regular $D$-class $D$ of $S$. One might think that these successive particularizations would result in some restriction on $D$, particularly if $\theta$ is onto; the object of this note is to show that this is not the case (Theorem 2 below).

Let $D$ be a regular $D$-class of $S^*$. Let

$$\{R_i^*: i^* \in I^*\} \quad \text{and} \quad \{L_\lambda^*: \lambda^* \in \Lambda^*\}$$

be the $R$-classes and $L$-classes, respectively, of $S^*$ contained in $D$. Then $H_i^* \cap L_\lambda^* = R_i^* \cap L_\lambda^*$ are the $3C$-classes of $S^*$ contained in $D$. We know that at least one of these must contain an idempotent, and so be a maximal subgroup of $S^*$ [2, Theorem 2.16, p. 59]; choose one such and call it $H^* = H_{i^* \lambda^*}^*$, $1^*$ being an element of $I^* \cap \Lambda^*$. For each $i^*$ in $I^*$, pick $r_{i^*}$ in $H_{i^*}^*$, and for each $\lambda^*$ in $\Lambda^*$ pick $q_{\lambda^*}$ in $H_{i^* \lambda^*}^*$. Then [2, Theorem 3.4, p. 92], every element of $D$ is uniquely representable in the form

$$(1) \quad r_{i^*}q_{\lambda^*} (x \in H^*; i^* \in I^*, \lambda^* \in \Lambda^*).$$

We regard the triple $(x; i^*, \lambda^*)$ as coordinates of the element (1).

By the Rees Theorem [2, Theorem 3.5, p. 94], a completely 0-simple semigroup can be represented as a regular Rees $I \times \Lambda$ matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$ over a group with zero $G^0$, and with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda^*})$. The elements of $\mathfrak{M}^0$ can be represented as triples $(a; i, \lambda)$ multiplying according to

$$(2) \quad (a; i, \lambda)(b; j, \mu) = (a\phi b; i, \mu) \quad (a, b \in G^0; i, j \in I; \lambda, \mu \in \Lambda).$$

In fact, the proof of the Rees Theorem amounts to coordinatizing the $3D$-class $\mathfrak{M}^0 \setminus 0$. It should be remarked that, for an arbitrary regular $3D$-class $D$, the elements (1) do not have a simple law of multiplication like (2).

**Theorem 1.** Let $S$ be a completely 0-simple semigroup represented as a regular Rees $I \times \Lambda$ matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$. Let $\theta$ be a partial homomorphism of $S$ into a semigroup $S^*$. Then $(S \setminus 0)^* \theta$ is contained in a regular $D$-class $D$ of $S$. Let $D$ be coordinatized as in (1). Then

$$\theta(a; i, \lambda) = r_{i^*}q_{\lambda^*} \phi(a; i, \lambda) (a \in G; i \in I, \lambda \in \Lambda),$$

where (i) $\phi: I \rightarrow I^*$ and $\psi: \Lambda \rightarrow \Lambda^*$ are mappings such that if $p_{\lambda^*} \neq 0$ then $q_{\lambda^*}r_{\lambda^*} \in H^*$;

(ii) $\omega: G \rightarrow H^*$ is a (group) homomorphism;
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(iii) \( u: I \rightarrow H^* \) and \( v: \Delta \rightarrow H^* \) are mappings such that if \( p_{\lambda i} \neq 0 \) then

\[
p_{\lambda i} \omega = v_{\lambda}(q_{\lambda i} r_{\lambda i}) u_{i}.
\]

The mappings \( \phi, \psi, \omega, u, v \) are uniquely determined by \( \theta \). Conversely, if mappings \( \phi, \psi, \omega, u, v \) are given satisfying (i), (ii), and (iii), then (3) defines a partial homomorphism \( \theta \) of \( S \backslash 0 \) into \( D \).

**Proof.** The proof is so much like that of Theorem 3.14 of [2, p. 109], that we give only the outline. We can assume that the entry \( p_{11} \) of \( P \) is not zero. The mappings \( \phi \) and \( \psi \) are determined by

\[
R_\lambda \subseteq R_{\theta \phi}, \quad L_\lambda \subseteq L_{\theta \phi},
\]

where \( \{R_i : i \in I\} \) are the \( R \)-classes, and \( \{L_\lambda : \lambda \in \Delta\} \) are the \( L \)-classes, of \( S \). This implies that

\[
(a; i, \lambda) \theta = r_{\phi x q_{\phi}}
\]

for some \( x \) in \( H^* \). If \( p_{\lambda i} \neq 0 \), then \( (p_{\phi x}^{-1}; i, \lambda) \theta \) is an idempotent in \( H_{\phi x, \lambda i} \), and it follows that \( q_{\phi x} r_{\phi i} \in H^* \) [2, Theorem 2.17, p. 59]. Defining \( \omega : G \rightarrow H^* \) by

\[
(p_{\phi x}^{-1}, a; 1, 1) \theta = r_{\phi x} h_{\phi x}^{-1}(a \omega) q_{\phi x}, \quad (h_0 = q_{\phi x} r_{\phi 1}),
\]

a brief calculation, using the uniqueness of the representation (1), shows that \( \omega \) is a homomorphism. For each \( i \in I \) and \( \lambda \in \Delta \) we define \( u_i \) and \( v_\lambda \) in \( H^* \) by

\[
(e; i, 1) \theta = r_{\phi x u_{i} q_{\phi x}},
\]

\[
(p_{\phi x}^{-1}, 1, \lambda) \theta = r_{\phi x} h_{\phi x}^{-1} v_{\lambda} q_{\phi x}.
\]

Applying \( \theta \) to

\[
(a; i, \lambda) = (e; i, 1)(p_{\phi x}^{-1}; a; 1, 1)(p_{\phi x}^{-1}; 1, \lambda),
\]

we obtain (3). Applying \( \theta \) to (2) and using (3), again with the uniqueness of (1), we obtain (4). This last step can be inverted to yield the converse part of the theorem.

From a constructive point of view, Theorem 1 has the drawback that, for given \( \phi, \psi, \) and \( \omega \) satisfying (i) and (ii), there is no assurance that \( u \) and \( v \) can be found so as to satisfy (iii). This drawback disappears, however, when \( S \) is a Brandt semigroup \( B^0 \). Here we can assume \( B^0 = \delta_{ij}(G; I, I; \Delta) \), where \( \Delta = (\delta_{ij}) \) is the \( I \times I \) identity matrix over \( G^0 \) [2, Theorem 3.9, p. 102]. The condition (4) now reduces to

\[
e^* = v_i q_{ij} r_{ij} u_i \quad (\text{all} \ i \in I),
\]
where \( e^* \) is the identity element of \( H^* \); or, what is equivalent, to

\[
(5) \quad v_i = u_i^{-1}(q_{i\phi}r_{i\phi})^{-1}.
\]

We note that \( q_{i\phi}r_{i\phi} \in H^* \) by (i). Thus we can always satisfy (iii) by choosing \( u: I \to H^* \) arbitrarily, and then defining \( v: I \to H^* \) by (5).

Formula (3) becomes

\[
(6) \quad (a; i, j)\theta = r_{i\phi}u_i(\omega\phi)u_j^{-1}(q_{i\phi}r_{i\phi})^{-1}q_{i\phi}.
\]

This differs from Hoehnke's formula (16) of [1, Part III, p. 97], chiefly because a definite coordinate system has been chosen for \( D \), independent of \( \theta \).

Now let \( D \) itself be a Brandt groupoid, say

\[
D = B^* = \mathfrak{M}(H^*; I^*, I^*; \Delta^*)\{0\}.
\]

Let us use square brackets to represent the elements \( [x^*; i^*, j^*] \) of \( B^* \). It is natural to choose \( r_{i^*} = [e^*; i^*, 1^*] \) and \( q_{i^*} = [e^*; 1^*, i^*] \). We then have \( q_{i^*}r_{i^*} = [e^*; 1^*, 1^*] \), while \( q_{i^*}r_{j^*} = 0 \) in \( B^{**} \), or is undefined in \( B^* \), if \( i^* \neq j^* \). Hence condition (i) of Theorem 1 requires that \( v_i^\phi = \phi \) for every \( i \) in \( I \); that is, \( \psi = \phi \). (5) becomes simply \( v_i = u_i^{-1} \), and (6) becomes

\[
(7) \quad (a; i, j)\phi = [u_i(\omega\phi)u_j^{-1}; i\phi, j\phi].
\]

Thus every partial homomorphism \( \theta \) of one Brandt groupoid, \( B \), into another, \( B^* \), is given by (7) in terms of (i) an arbitrary mapping \( \phi: I \to I^* \), (ii) an arbitrary homomorphism \( \omega: G \to H^* \), and (iii) an arbitrary mapping \( u: I \to H^* \). (7) is equivalent to Hoehnke's formula (22) in [1, Part I, p. 164]. It can also be obtained by specialization from Theorem 3.14 of [2].

2. Partial homomorphic images of Brandt groupoids. We come now to the main result of the present note.

**Theorem 2.** Any regular \( \mathcal{D} \)-class of any semigroup is a partial homomorphic image of some Brandt groupoid.

**Proof.** Let \( D \) be a regular \( \mathcal{D} \)-class of a semigroup \( S \). Let

\[
\{ R_i: i \in I \} \quad \text{and} \quad \{ L_\lambda: \lambda \in \Lambda \}
\]

be the \( \mathcal{R} \)-classes and \( \mathcal{L} \)-classes, respectively, of \( S \) contained in \( D \). As usual, we may assume that \( I \) and \( \Lambda \) have an element 1 in common such that \( H_{11} = R_1 \cap L_1 \) is a group. But now we shall also assume, as we evidently may, that \( I \) and \( \Lambda \) are otherwise disjoint: \( I \cap \Lambda = \{ 1 \} \).

As usual, choose \( r_i \) in \( H_{1i} \) and \( q_i \) in \( H_{1i} \) in any way, for \( i \) in \( I \setminus 1 \) and \( \lambda \) in \( \Lambda \setminus 1 \), and choose \( r_1 = q_1 = e_{11} \), the identity element of \( H_{11} \). As
in (1), without the stars, every element of \( D \) is uniquely representable in the form

\[
(r_i a) q_{i\lambda} \quad (a \in H_{1i}; \ i \in I, \ \lambda \in \Lambda).
\]

For \( i \) in \( I \setminus 1 \) and \( \lambda \) in \( \Lambda \setminus 1 \), let \( q_i \) be any inverse of \( r_i \) in \( R_i \), and let \( r_{i\lambda} \) be any inverse of \( q_{i\lambda} \) in \( L_i \). Then

\[
q_{i\lambda} r_{i\lambda} = e_1 \quad (\text{all } \alpha \text{ in } I \cup \Lambda).
\]

Let \( B = \mathfrak{M}(H_{11}; I \cup \Lambda, I \cup \Lambda; \Delta) \setminus 0 \). Denote the elements of \( B \) by triples \((a; \alpha, \beta)\). Multiplication in \( B \) is given by

\[
(a; \alpha, \beta)(b; \beta, \gamma) = (ab; \alpha, \gamma) \quad (a, b \in H_{11}; \alpha, \beta, \gamma \in I \cup \Lambda).
\]

Products \((a; \alpha, \beta)(b; \beta', \gamma)\) with \( \beta \neq \beta' \) are not defined in \( B \) (and are zero in \( B^0 \)). Define \( \theta: B \to D \) as follows:

\[
(a; \alpha, \beta) \theta = r_{a} a q_{\beta} \quad (a \in H_{11}; \alpha, \beta \in I \cup \Lambda).
\]

Then, because of (9),

\[
(a; \alpha, \beta) \theta(b; \beta, \gamma) \theta = r_{a} a q_{\beta} r_{\beta} b q_{\gamma} = r_{a} ab q_{\gamma} = (ab; \alpha, \gamma) \theta.
\]

From this and (10), it follows that \( \theta \) is a partial homomorphism of \( B \) into \( D \). Moreover, \( B \theta = D \), since \( B \theta \) contains all the elements \( r_{a} a q_{\lambda} \) of \((8)\).

As described in §3.3 of [2], if we adjoin a zero element \( 0 \) to a Brandt groupoid \( B \), defining \( ab = 0 \) if \( ab \) is undefined in \( B \), we obtain a Brandt semigroup \( B^0 \), that is, a completely 0-simple inverse semigroup. The following is immediate from Theorem 2 and the first assertion in Theorem 1.

**Corollary 1.** A semigroup with zero is a partial homomorphic image of some Brandt semigroup if and only if it is regular and 0-bisimple.

As defined in [2, p. 93], a *partial isomorphism* is a partial homomorphism which is one-to-one and onto. Not every regular \( \mathcal{D} \)-class is a partial isomorphic image of some Brandt groupoid, and the question of telling which ones are remains unsettled. The next theorem gives a sufficient condition.

**Theorem 3.** Let \( D \) be a regular \( \mathcal{D} \)-class of a semigroup \( S \) with the property that it is possible to set up a one-to-one correspondence between the \( \mathcal{R} \)-classes \( R \) of \( D \) and the \( \mathcal{L} \)-classes \( L \) of \( D \) such that if \( R \) and \( L \) correspond, then \( RL \) contains an idempotent. Then \( D \) is a partial isomorphic image of the Brandt groupoid having the same structure group as \( D \) and the same number of \( \mathcal{R} \)-classes (and \( \mathcal{L} \)-classes) as \( D \).
Proof. By hypothesis, we can index the $\beta$-classes and the $\xi$-classes of $D$ by the same index set $I$, such that for each $i$ in $I$, $R_i \cap L_i$ contains an idempotent $e_i$. The $3C$-class $H_{ii} = R_i \cap L_i$ is then the maximal subgroup $H_{ii}$ of $S$ containing $e_i$. Let $1 \in I$, and pick $q_i$ in $H_{ii}$ in any way for $i$ in $I \setminus 1$, and let $q_i = e_i$. Let $q'_i$ be the inverse of $q_i$ in $H_{ii}$; such exists since both $H_{ii}$ and $H_{ii}$ contain idempotents [2, Theorem 2.18, p. 60]. Take $B = \mathfrak{M}^\theta(H_{ii}; I, I; \Delta) \setminus 0$ and define $\theta: B \to D$ by

$$\theta(a; i, j) = q'_i a q_j \quad (a \in H_{ii}; i, j \in I).$$

Since every element of $D$ is uniquely expressible in the form on the right-hand side of (11), and $q_i q'_i = e_i$, we see at once that $\theta$ is a partial isomorphism of $B$ onto $D$.

$B$ is unique, to within isomorphism, since any Brandt groupoid is completely determined by its structure group and the cardinal number of its $\beta$-classes (or $\xi$-classes).

Corollary 2. Every $0$-bisimple inverse semigroup $S$ is a partial isomorphic image of the Brandt semigroup having the same structure group as $S$ and the same number of idempotents as $S$.

Proof. The hypothesis of Theorem 3 is satisfied by any inverse semigroup [2, Corollary 2.19, p. 60]. For $0$-bisimple inverse semigroups, in particular, for Brandt semigroups, the sets of $\beta$-classes, $\xi$-classes, and nonzero idempotents all have the same cardinal.

We conclude with an example to show that a regular $0$-bisimple semigroup may be a partial isomorphic image of a Brandt semigroup, but not of one having the same structure group.

Let $B = \mathfrak{M}^\theta(E; I, I; \Delta) \setminus 0$, where $E = \{e\}$ is a one-element group, and $I = \{1, 2\}$. Let $S \setminus 0 = H \times E$, where $H$ is a cyclic group $\{e, a\}$ of order 2, and $E$ is a right zero semigroup of order 2. We may represent the elements of $S$ as pairs $(x; i)$ with $x \in H$, $i \in I$, multiplying as follows:

$$(x; i)(y; j) = (xy; j) \quad (x, y \in H; i, j \in I).$$

Define $\theta: B \to S$ by

$$\theta(e; 1, 1) = (e; 1), \quad (e; 1, 2) = (a; 2)$$
$$\theta(e; 1, 1) = (e; 1), \quad (e; 2, 1) = (a; 2)$$
$0 \theta = 0$. Clearly $\theta$ is one-to-one and onto, and it is easy to verify that it is a partial homomorphism.

On the other hand, $S$ cannot be a partial isomorphic image of any Brandt semigroup $B$ having structure group of order 2. For $B$ must then have order twice a square, whereas $S \setminus 0$ has order 4.
EVERY STANDARD CONSTRUCTION IS INDUCED BY A PAIR OF ADJOINT FUNCTORS

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In this note, we prove the converse of the following result of P. Huber [2]. Let \( F : \mathcal{K} \rightarrow \mathcal{L} \) and \( G : \mathcal{L} \rightarrow \mathcal{K} \) be covariant \textit{adjoint functors}, that is, functors such that there exist two (functor) morphisms \( \xi : I \rightarrow GF \) and \( \eta : FG \rightarrow I \) satisfying the relations

\begin{align*}
(1) & \quad (\eta \ast F) \circ (F \ast \xi) = 1 \ast F, \\
(2) & \quad (G \ast \eta) \circ (\xi \ast G) = 1 \ast G.
\end{align*}

Then, the triple \((C, k, p)\) given by

\[
C = FG, \quad k = \eta \quad \text{and} \quad p = F \ast \xi \ast G
\]

is a \textit{standard construction} in \( \mathcal{L} \), that is, \( C \) is a covariant functor, \( k : C \rightarrow I \) and \( p : C \rightarrow C^2 \) are (functor) morphisms, and the following relations hold:

\begin{align*}
(3) & \quad (k \ast C) \circ p = (C \ast k) \circ p = 1 \ast C, \\
(4) & \quad (p \ast C) \circ p = (C \ast p) \circ p.
\end{align*}

This standard construction is said to be \textit{induced by the pair of adjoint functors} \( F \) and \( G \). For further explanation of the notation and terminology, see [2], or the appendix of [1].

\textbf{Theorem.} Let \((C, k, p)\) be a standard construction in a category \( \mathcal{L} \). Then there exists a category \( \mathcal{K} \) and two covariant functors \( F : \mathcal{K} \rightarrow \mathcal{L} \) and \( G : \mathcal{L} \rightarrow \mathcal{K} \) such that

\begin{enumerate}
  \item \( F \) is (left) adjoint to \( G \),
  \item \((C, k, p)\) is induced by \( F \) and \( G \).
\end{enumerate}

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