References

1. E. F. Beckenbach and R. Bellman, *Inequalities*, Chapter 2, Section 8, Springer, Berlin, 1961.

U. S. Naval Ordnance Laboratory[2]

[2] The author is on leave of absence and is presently at the Weizmann Institute.

NOTE ON DIFFERENTIAL OPERATORS WITH A PURELY CONTINUOUS SPECTRUM

F. ODEH

In [1], Kreith gave an example of a Sturm-Liouville operator with positive coefficients,

$$Lu = -\frac{1}{r(x)}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u, \qquad 0 \leqq x < \infty,$$
$$u(0) = 0,$$

which has a purely continuous spectrum. The novelty of the example lies in the relatively weak assumptions on the potential q. Thus, in the case $p=r=1$, one need not assume that q is integrable at infinity —compare [2, Chapter 9, Problem 4]—but it is sufficient to assume q to be monotonically decreasing. In this note a similar theorem is given which holds in any number of dimensions. The proof, which applies to Kreith's case also, shows that the nonexistence of eigenfunctions may be ascribed to two different reasons depending on the asymptotic behavior of $q(x)$. In one simple case it is due to the boundary condition while in the other, and more important, case it is a consequence of the behavior of q at infinity. For simplicity the proof is restricted to the case of Schroedinger's equation in three dimensions, defined in the exterior X of a closed smooth surface Γ. Hence, we consider the eigenvalue problem,

$$Lu = -\Delta u + qu = \lambda u, \tag{1a}$$

subject to the boundary conditions

Received by the editors December 18, 1963.

$$u \quad \text{or} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{1b}$$

The potential q is assumed to be real, continuous, and tends uniformly to a limit q_∞. Then the boundary-value problem (1a, b) gives rise to a self-adjoint operator in $L_2(X)$, which we denote again by L. We first discuss a special case in which the boundary conditions (1b) play no role. Let r denote the radial variable $|x|$; then one has the following

THEOREM. *If $\partial q/\partial r \leqq 0$ for large enough r and $q_\infty < \lambda$, then there is no L_2-solution to equation* (1a) *different from zero.*

The proof of the theorem follows very closely the technique of Kato [3] and we introduce the same notation. Let $k^2 = \lambda - q_\infty$ and $p(x) = q(x) - q_\infty$. By the unique continuation theorem (see Calderon [4]), it is sufficient to prove that all L_2-solutions to

$$(\Delta + k^2 - p(x))u = 0, \tag{2}$$

where $p(x) \to 0$ as $r \to \infty$ and $\partial p/\partial r \leqq 0$ for large r, must vanish identically near infinity. Now, the solution of (2) may be regarded as a vector, depending on a parameter r, in the Hilbert space $H = L_2(\Omega)$, where Ω is the unit sphere, with the usual definition of scalar product. If $v_m = r^{m+1}u$, $v = v_0 = ru$, then, from (2), v_m satisfies the ordinary differential equation

$$v_m'' - \frac{2m}{r} v_m' + \frac{1}{r^2}[m(m+1) - A]v_m + (k^2 - P)v_m = 0, \tag{3}$$

where A is the negative of the Laplace-Beltrami operator on Ω and $P(r)$ is the operator defined in H through multiplication by $p(x)$ for $|x| = r$. One now defines an "energy flux" across the sphere $|x| = r$ by the relations

$$\begin{aligned} F(m, t; r) = {} & \|v_m'\|^2 + [k^2 - 2ktr^{-1} + m(m+1)r^{-2}]\|v_m\|^2 \\ & - (v_m, Pv_m) - \frac{1}{r^2}(Av_m, v_m), \quad m, t > 0, \end{aligned} \tag{4}$$

$$F(r) = F(0, 0; r) = \|v'\|^2 + (v, (k^2 - P)v) - \frac{1}{r^2}(Av, v). \tag{5}$$

The following two lemmas are concerned with the growth properties of F.

LEMMA 1. *If r is large enough, then $F'(r) \geqq 0$.*

PROOF. Differentiating (5) and using the symmetry of A, we get

$$F'(r) = 2\left\{\left(v', v'' + (k^2 - P)v - \frac{1}{r^2} Av\right)\right\} + \frac{2}{r^3}(Av, v) + (k^2 - P)'\|v\|^2$$
$$\geqq (k^2 - P)'\|v\|^2,$$

where we have used the fact that A is positive together with equation (3). The last expression is nonnegative for large r by hypothesis.

LEMMA 2. *If $u \in L_2$ does not vanish identically, then there are arbitrarily large values of r for which $F(r) > 0$.*

The proof follows exactly the same lines as Kato [3, Lemmas 1, 2, 3]. One has only to substitute the new definitions of F, $F(m, t; r)$ and to take into account the fact that $\partial p/\partial r \leqq 0$ eventually. Since no other new ideas are involved, we do not reproduce Kato's procedure here.

To complete the proof of the theorem, one can proceed in either of two ways.

(i) From Lemmas 1, 2, there exists $R > 0$ and $\delta > 0$ so that

$$\int_R^\rho \left\{\|v'\|^2 + (v, (k^2 - P)v) - \frac{1}{r^2}(Av, v)\right\} dr \geqq \delta\rho \quad \text{if } \rho \text{ is large enough.} \tag{6}$$

On the other hand, by scalar multiplying (3), for $m = 0$, by v and integrating, one gets

$$\begin{aligned} 0 &= \int_R^\rho \left(v, v'' + (k^2 - P)v - \frac{1}{r^2} Av\right) dr \\ &= \int_R^\rho \left(v, (k^2 - P)v - \frac{1}{r^2} Av\right) dr - \int_R^\rho \|v'\|^2\, dr + (v, v')\Big|_R^\rho . \end{aligned} \tag{7}$$

Now choose a sequence $\rho_n \to \infty$ so that $(v, v') \leqq 0$—this is always possible since $u \in L_2$—then from (6), (7) there exists $\eta > 0$ such that

$$\int_R^{\rho_n} \left(v, (k^2 - P)v - \frac{1}{r^2} Av\right) dr \geqq \eta\rho_n. \tag{8}$$

Since A is nonnegative and R may be chosen large enough so that P also is nonnegative, one gets

$$\int_R^{\rho_n} \|v\|^2\, dr \geqq \eta\rho_n.$$

Since $v = ru$, this implies that u does not belong to L_2.

(ii) From Lemmas 1, 2 one has

$$\liminf_{r\to\infty}\{\|v'\|^2 + k^2\|v\|^2\} > 0. \tag{9}$$

But if $u\in L_2$ and $(\Delta+k^2-p)u=0$, then $\nabla u\in L_2$, as is easily seen from Green's formula and the finiteness of q. The growth estimate (9) then leads to an immediate contradiction.

In order to give an example where no eigenfunctions occur, we first need a simple lemma in which the form of the boundary condition on Γ plays an important role.

LEMMA 3. *If* $q(x)-\lambda\geqq 0$, *then the only* L_2*-solution of* (1a, b) *is* $u\equiv 0$.

The proof is rather obvious.

To obtain an operator with purely continuous spectrum, one may assume that the potential q is always positive, tends uniformly to a limit and that $\partial q(x)/\partial r\leqq 0$ for $x\in X$. Then it is sufficient to consider $\lambda\geqq 0$ and either the hypotheses of Lemma 3 or the hypotheses of the theorem will be satisfied; hence no eigenfunctions would exist. We remark here that the monotonic decreasing property assumed in the theorem cannot be dropped, as shown by an example of von Neumann and Wigner [5].

REFERENCES

1. K. Kreith, *Differential operators with a purely continuous spectrum*, Proc. Amer. Math. Soc. **14** (1963), 809–811.

2. E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

3. T. Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Comm. Pure Appl. Math. **12** (1959), 403–425.

4. A. P. Calderon, *Uniqueness in the cauchy problem for partial differential equations*, Amer. J. Math. **80** (1958), 16–36.

5. J. von Neumann and E. P. Wigner, *Über merkwürdige diskrete Eigenwerte*, Z. Physik **50** (1929), 465–467.

INTERNATIONAL BUSINESS MACHINES CORPORATION