ON INTEGRAL TRANSFORMATIONS WITH
POSITIVE KERNEL

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1. Let $X, Y$ be two totally σ-finite measure spaces and let $F_X, F_Y$ be
the classes of measurable functions defined, and finite a.e., in $X, Y,
respectively.

Let $K(x, y) \geq 0$ be a measurable non-negative function on $X \times Y$.$
Let us denote by $T, T^*$ the transformations:

$$(Tu)(y) = \int_X K(x, y)u(x)dx, \quad (T^*v)(x) = \int_Y K(x, y)v(y)dy,$$

the domain of $T$ being the set of the functions $u(x) \in F_X$ such that
the first integral exists and is finite for almost all $y$, and the domain
of $T^*$ being analogously defined.

We are interested in the study of necessary and sufficient conditions in order that $T$ restricted to $L^p(X)$ be a bounded transformation
of $L^p(X)$ into $L^q(Y)$ with bound $\leq c_0$.

Obviously such a necessary and sufficient condition is given by:

$$\left(1.1\right) \quad \int_{X \times Y} K(x, y)u(x)v(y) \, dx \, dy \leq c_0 \|u\|_{L^p(X)} \|v\|_{L^q(Y)},$$

\forall u \in L^p(X), \forall v \in L^q(Y).

In the present paper we will prove the necessity of the following
sufficient condition:

[1.1] Theorem (N. Aronszajn\(^2\)). Let $1 < q \leq p < + \infty$, $1/p' + 1/p = 1/q' + 1/q = 1$. A sufficient condition in order that $T$ restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leq c_0$ is that for every $\varepsilon > 0$ there exist two measurable functions $\phi(x), \psi(y)$, positive and finite a.e., such that:

$$\left(1.2\right) \quad (T\phi)(y) \leq (c_0 + \varepsilon)(\psi(y))^{q'/q},$$

$$(T^*\psi)(x) \leq (c_0 + \varepsilon)(\phi(x))^{p'/p}.$$
no other condition for $p = q$,

$$\int \int_{x \times y} K(x, y) \phi(x) \psi(y) \, dx \, dy \leq c_0 + \varepsilon \quad \text{for } p \neq q. \tag{1.3}$$

In §3 we will prove the following:

[1.II] Theorem. Let $1 < p < + \infty$, $1 < q < + \infty$.

A necessary condition in order that $T$ restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leq c_0$ is given by (1.2) with $\phi, \psi$, positive and finite a.e., satisfying:

$$\|\phi\|_{L^p(X)} \leq 1, \quad \|\psi\|_{L^q(Y)} \leq 1. \tag{1.4}$$

This theorem in the case $p = q = 2$ is due to S. Karlin [4].

[1.III] Corollary. Obviously (1.2)--(1.3) is weaker than (1.2)--(1.4), and, consequently, for $q \leq p$ both (1.2)--(1.3) and (1.2)--(1.4), a posteriori equivalent, are necessary and sufficient.

[1.IV] Remark. For $q > p$, as we will show by an example, even the stronger condition (1.2)--(1.4) is in general not sufficient.

[1.V] Remark. For $\varepsilon = 0$ Theorem [1.I] does not hold.

[1.VI] Remarks. For $q = 1$, $p$ finite or not, a necessary and sufficient condition is: $\|\int_K K(x, y) \psi(x) \, dy\|_{L^p(X)} \leq c_0$. For $p = + \infty$, $q$ finite or not, a necessary and sufficient condition is: $\|\int_K K(x, y) \phi(x) \, dx\|_{L^q(Y)} \leq c_0$.

2. Proof of Theorem [1.I].

Case $p \neq q$. Given $u \in L^p(X)$, $v \in L^q(Y)$, we note that:

$$\left| \int_X \int_Y K(x, y) u(x) v(y) \, dx \, dy \right|$$

$$= \left| \int_X \int_Y K^{1/q-1/p} \phi^{1/q-1/p} \psi^{1/q-1/p} \frac{\psi^{1/p}}{\phi^{1/p}} \frac{\phi^{1/q'}}{\psi^{1/q'}} u \, dx \, dy \right|$$

$$\leq \left( \int_X \int_Y K(x, y) \phi(x) \psi(y) \, dx \, dy \right)^{1/q-1/p}$$

$$\cdot \left( \int_X \int_Y K(x, y) \frac{\psi}{\phi^{1/p'}} \, |u|^p \, dx \, dy \right)^{1/p}$$

$$\cdot \left( \int_X \int_Y K(x, y) \frac{\phi}{\psi^{1/q'}} \, |v|^{q'} \, dx \, dy \right)^{1/q'}$$

$$\leq (c_0 + \varepsilon) \|u\|_{L^p(X)} \|v\|_{L^q(Y)}.$$

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* In the following meaning: functions $\phi, \psi$ satisfying (1.2), (1.4) satisfy also (1.3), and conversely if there exist $\phi, \psi$ satisfying (1.2), (1.3) then there exist also $\bar{\phi}, \bar{\psi}$ (may be different) satisfying (1.2), (1.4).
Case $p = q$. Since $1/q - 1/p = 0$ the proof here given does not require the boundedness of the integral in (1.3).

3. Proof of Theorem [1.II].

[3.1] Lemma. Let $B$ be a Banach space and $P$ a convex cone in $B$. By calling this cone "positive," $B$ will be taken as an ordered Banach space. Let us suppose for $B$ and $P$ that every bounded increasing sequence in $P$ converges, more precisely:

(3.1) $\{u_n\} \subset P$, $u_{n+1} - u_n \in P$, $\|u_n\| \leq M < + \infty \Rightarrow u_n \to u \in P$.

Let $S$ be a (not necessarily linear) transformation defined in $P$ such that

$$S(P) \subset P,$$

(3.2) $S$ is nondecreasing: $u, v, v - u \in P \Rightarrow Sv - Su \in P,$

$S$ is continuous,$^4$

$$\|u\| \leq 1 \Rightarrow \|Su\| \leq 1.$$

Then for every $\sigma > 0$ there is $u \in P$ such that:

(3.3) $(1 + \sigma)u - Su \in P$,

$$\|u\| \leq 1, \quad u \neq 0.$$

Proof of lemma. Choose $p \in P$, $p \neq 0$, $\|p\| \leq \sigma/(1 + \sigma)$ and consider the sequence $\{u_n\} \subset P$ defined as follows:

$$u_1 = p, \quad u_n = p + \frac{1}{1 + \sigma} Su_{n-1}.$$

Obviously by induction,

$$\|u_n\| \leq 1, \quad u_{n+1} - u_n \in P.$$

So by (3.1), and since $S$ is continuous:

$$u_n \to u \in P, \quad \|u\| \leq 1, \quad u = p + \frac{1}{1 + \sigma} Su.$$

Hence also:

$$u \neq 0, \quad (1 + \sigma)u - Su = (1 + \sigma)p \in P.$$

The necessity of (1.2), (1.4) follows from Lemma [3.1] by taking $B = L^p(X)$, $u \in P \iff u(x) \geq 0$ a.e.,

$^4$ In the strong topology, or, more generally, relative to any topology for which (3.1) holds, and the norm is lower semicontinuous.
where
\[
\psi_0(y) > 0 \text{ a.e., } \|\psi_0\|_{L^q} \leq \frac{\epsilon}{c_0 + \epsilon},
\]
\[
\psi_0(x) > 0 \text{ a.e., } \|\psi_0\|_{L^p} \leq \frac{\sigma}{1 + \sigma},
\]
and \(\sigma\) is such that
\[
\sigma (1 + \sigma)^{2p/p'} \leq c_0 + \epsilon.
\]
For, in the present case, (3.3) becomes
\[
\left( \frac{1}{c_0} T^\# \left( v_0 + \frac{1}{c_0 + \epsilon} T \left( \psi_0 + \frac{1}{1 + \sigma} \psi \right) \right)^{q/p'} \right)^{p/q'} \leq (1 + \sigma) \psi
\]
\[
\|\psi\|_{L^p(x)} \leq 1,
\]
and by putting
\[
\phi = \psi_0 + \frac{1}{1 + \sigma} \psi, \quad \psi = \left( v_0 + \frac{1}{c_0 + \epsilon} T \phi \right)^{q/q'},
\]
we note that:
\[
T \phi = (c_0 + \epsilon) (\psi^{q/q'} - v_0) \leq (c_0 + \epsilon) \psi^{q/q'},
\]
\[
T^\# \psi \leq c_0 ((1 + \sigma) \psi)^{p/p'} = c_0 (1 + \sigma)^{2p/p'} (\phi - u_0)^{p/p'} \leq (c_0 + \epsilon) \phi^{p/p'},
\]
\[
\|\phi\|_{L^p(x)} \leq \frac{\sigma}{1 + \sigma} + \frac{1}{1 + \sigma} = 1, \quad \phi \geq u_0 > 0 \text{ a.e.},
\]
\[
\|\psi\|_{L^p(x)} \leq \left( \frac{\epsilon}{c_0 + \epsilon} + \frac{c_0}{c_0 + \epsilon} \right)^{q/q'} = 1, \quad \psi \geq (v_0)^{q/q'} > 0 \text{ a.e.}
\]

4. Proof of Remark [1.IV]. Consider the following example:

\(p = 2, \quad (p' = 2), \quad q = 4, \quad (q' = \frac{4}{3}), \quad X = (0, +\infty), \quad Y = (0, 1),\)

\[
K(x, y) = \begin{cases} 
\frac{1}{y} & \text{for } \frac{y}{2} < x < y, \\
0 & \text{elsewhere,}
\end{cases}
\]
\[ \phi(x) = \frac{1}{\sqrt{2}} \frac{1}{(x + 1)^{3/4}} \quad (0 < x < +\infty), \]
\[ \psi(y) = \frac{1}{\sqrt{2}} \frac{1}{(y + 1)^{3/4}} \quad (0 < y < 1). \]

To prove the first inequality in (1.2) (with some suitable \( c_0 + \epsilon \)) it is enough to note that \((T\phi)(y)\) is bounded, and \(\psi(y)\) has a positive infimum, and to prove the second inequality it is enough to note that \((T^*\psi)(x)\) is bounded, and is \(\equiv 0\) for \(x \geq 1\).

Conditions (1.4) are immediately verified.

Finally, to prove that \(T\) is not, however, a bounded transformation of \(L^2(0, +\infty)\) into \(L^4(0, 1)\) it is enough to consider the function:
\[
\begin{aligned}
f(x) &= \begin{cases} 
\frac{1}{x^{1/4}} & \text{for } 0 < x < 1, \\
0 & \text{elsewhere,}
\end{cases}
\end{aligned}
\]

which belongs to \(L^2(0, +\infty)\), while \((Tf)(y) \in L^4(0, 1)\).

5. Proof of Remark [1.V]. Let us take \(p=q=p'=q'=2, X=Y=\text{x-axis}, \]
\[
K(x, y) = K(y, x) = \begin{cases} 
1 & \text{for } |x - y| \leq \frac{1}{2}, \\
0 & \text{elsewhere,}
\end{cases}
\]

We will prove that there are no non-negative \(\phi, \psi, \phi(x) \neq 0 \neq \psi(x)\), such that \(\phi \in L^2(X)\), \(T\phi \leq \psi\), \(T\psi \leq \phi\).

Let us suppose that such \(\phi, \psi\) exist. Putting \(u(x) = \phi(x) + \psi(x)\), we have
\[
u(x) \geq 0, \quad u(x) \neq 0, \quad Tu \leq u, \quad Tu \leq T\phi + \psi \in L^2.
\]

**First case.** Let us suppose \(Tu = u\) a.e. This case can be easily excluded (for instance by taking Fourier transforms of both sides.)

**Second case.** \(Tu < u\) on a set of positive measure. Then we can choose \(c_1, \alpha_1\) such that:
\[
\int_{-\alpha}^{\alpha} u(x) \, dx - \int_{-\alpha}^{\alpha} Tu \, dx \geq c_1 > 0 \quad \text{for } \alpha \geq \alpha_1.
\]

On the other hand since \(Tu \in L^2\) there is a set \(I\) of infinite measure such that
\[
\int_{-\alpha-1/2}^{\alpha+1/2} u(x) \, dx + \int_{-\alpha+1/2}^{\alpha-1/2} u(x) \, dx = (Tu)(-\alpha) + (Tu)(\alpha) < c_1 \quad \text{for } \alpha \in I.
\]
From the definition of $T$ it follows that for every $\alpha > 0$,

$$\int_{-a}^{a} u(x) \, dx - \int_{-a}^{a} Tu \, dx \leq \int_{-a-1/2}^{-a+1/2} u(x) \, dx + \int_{a-1/2}^{a+1/2} u(x) \, dx,$$

and if we take $\alpha \geq \alpha_1$, $\alpha \in I$ we have a contradiction.

6. Proof of Remarks [1.VI].

First Remark: Sufficiency. Given $u \in L^p(X)$, $v \in L^q(Y)$ we have

$$\left| \int_X \int_Y K(x, y) u(x) v(y) \, dxdy \right| \leq \|v\|_{L^q(Y)} \int_X \int_Y |K(x, y)| u(x) \, dxdy \leq c_0 \||v\|_{L^q(Y)} \|u\|_{L^p(X)} \|v\|_{L^q(Y)}.$$

Necessity. Since for every $u \in L^p(X)$, $v \in L^q(Y)$ we have

$$\int_X \int_Y K(x, y) u(x) v(y) \, dxdy \leq c_0 \|u\|_{L^p(X)} \|v\|_{L^q(Y)},$$

taking $v \equiv 1$ we have

$$\int_X u(x) \left( \int_Y K(x, y) \, dy \right) \, dx \leq c_0 \|u\|_{L^p(X)},$$

and since $u \in L^p(X)$ is still arbitrary:

$$\left\| \int_Y K(x, y) \, dy \right\|_{L^q(X)} \leq c_0.$$

Second Remark. The proof is analogous.

References

2. T. Carleman, Sur les équations intégrales singulières à noyau réel et symétrique, Upsala, 1923.

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