ON A CLASS OF COUNTABLY PARACOMPACT SPACES

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In this note we shall characterize a topological property which is stronger than countable paracompactness but which is equivalent to it for normal spaces.

A real valued function on a topological space $X$ is locally bounded if each point has a neighborhood on which the function is bounded. Let $C(X)$ denote the set of real valued continuous functions on $X$. A topological space is a cb-space if for each locally bounded function $h$, there exists $f \in C(X)$ such that $f \geq |h|$. J. G. Horne, Jr. initiated a study of cb-spaces and reported on his work in [2].

For $f \in C(X)$, the set on which $f$ vanishes is called the zero-set of $f$ and it is denoted by $Z(f)$. The complement of a zero-set is called a cozero-set. A cozero cover is a cover consisting of cozero-sets. A family of continuous functions is locally finite if the collection of cozero-sets associated with the family is a locally finite collection of sets. A family $\mathcal{F}$ of continuous functions is a partition of unity if $0 \leq f$ for all $f \in \mathcal{F}$ and $\sum_{f \in \mathcal{F}} f(x) = 1$ for all $x \in X$. A partition of unity is subordinate to a cover if the collection of cozero-sets associated with the partition is a refinement of the cover. A countable cover $\{U_n\}$ is an increasing cover if $U_n \subset U_{n+1}$ for all $n$. In this paper the term cover will be used to mean open cover.

**Theorem 1.** For any topological space $X$, the following statements are equivalent:

(a) $X$ is a cb-space.

(b) Given an upper-semicontinuous function $h$ on $X$, there exists $f \in C(X)$ such that $f \geq h$.

(c) Given a positive (nonvanishing) lower-semicontinuous function $g$ on $X$, there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$.

(d) For each countable increasing cover of $X$, there exists a locally finite partition of unity subordinate to that cover.

(e) For each countable increasing cover of $X$, there exists a partition of unity subordinate to that cover.

(f) Each countable increasing cover of $X$ has a locally finite cozero refinement.

(g) Each countable increasing cover of $X$ has a $\sigma$-locally finite cozero refinement.

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(h) Each countable increasing cover of $X$ has a countable cozero refinement.

(i) Given a decreasing sequence $\{F_n\}$ of closed sets in $X$ with empty intersection, there exists a sequence $\{Z_n\}$ of zero-sets with empty intersection such that $Z_n \supset F_n$.

Proof. (a) $\rightarrow$ (b). If $h$ is upper-semicontinuous, then $h^+$ is locally bounded. Thus there exists $f \in C(X)$ such that $f \geq h^+ \geq h$.

(b) $\rightarrow$ (c). If $g$ is strictly positive and lower-semicontinuous, then $g^{-1}$ exists and is upper-semicontinuous. Let $\phi \in C(X)$ be such that $\phi \geq g^{-1}$. Then $f = \phi^{-1} \in C(X)$ and $0 < f(x) \leq g(x)$ for all $x \in X$.

(c) $\rightarrow$ (d). For a countable increasing cover $\{U_n\}$, define $g(x) = 1$ on $U_1$ and $g(x) = n^{-1}$ on $U_n - U_{n-1}$ for $n > 1$. Then $g$ is lower-semicontinuous and strictly positive. For $f \in C(X)$ such that $0 < f(x) \leq g(x)$ on $X$, define $\phi_n = [(n+1)f-1]^+ - [(n-1)f-1]^{-}$ and $\phi = \sum \phi_n$ (here $1(x) = 1$ for all $x \in X$). Since $\{\phi_n\}$ is locally finite, $\phi \in C(X)$. Furthermore, $\phi$ is nonvanishing; thus $\phi$ is a unit of the ring $C(X)$. It now follows that $\{\phi^{-1}\phi_n\}$ is a locally finite partition of unity. That this partition is subordinate to $\{U_n\}$ is a consequence of the following: $X - Z(\phi^{-1}\phi_n) = \{x: (n+1)^{-1} < f(x) < (n-1)^{-1}\} \subset \{x: (n+1)^{-1} < g(x)\} = U_n$.

Statements (d), (e), and (f) are equivalent [6, 1.2]. Clearly (f) implies (g).

(g) $\rightarrow$ (h). Let $\mathcal{U}$ be a $\sigma$-locally finite cozero refinement of the increasing cover $\{U_n\}$. Then $\mathcal{U} = \bigcup \mathcal{V}_m$ where each $\mathcal{V}_m$ is a locally finite collection of cozero-sets. Let $V_m$ denote the union of the sets $V$ in $\mathcal{V}_m$ such that $V \subset U_n$. Since $V_m$ is the union of a locally finite collection of cozero-sets, it is a cozero-set. Thus $\{V_m\}$ is a countable cozero refinement of $\{U_n\}$.

(h) $\rightarrow$ (i). If $\{F_n\}$ is a decreasing sequence of closed sets for which $\bigcap F_n$ is empty, then $\{X - F_n\}$ is an increasing cover of $X$. Let $A$ be a countable subset of $C(X)$ such that $\{X - Z(f)\}_{f \in A}$ is a refinement of $\{X - F_n\}$. Denote by $Z_n$ the intersection of those $Z(f)$, $f \in A$ for which $Z(f) \supset F_n$. Each $Z_n$ is a countable intersection of zero-sets and hence is a zero-set [3, §1.14]. Now $\bigcap Z_n = \bigcap_{f \in A} Z(f)$ is vacuous. Whence $\{Z_n\}$ is the desired sequence.

(i) $\rightarrow$ (a). Let $h$ be a locally bounded function on $X$ and define $F_n = \overline{\{x: |h(x)| \geq n\}}$. Then $\{F_n\}$ is a decreasing sequence of closed sets. Since $h$ is locally bounded, this sequence has empty intersection. Thus by (i) there exists a sequence $\{g_n\}$ in $C(X)$ such that $Z(g_n) \supset F_n$ and $\bigcap Z(g_n)$ is empty. Define $f_n = 1 - \bigvee_{g_n} n|g_i| \wedge 1$ where $1(x) = 1$ for all $x \in X$. Given $x \in X$, there exists $i$ such that $g_i(x) \neq 0$; whence there
exists a positive integer $j$ such that $|g_i(x)| > j^{-1}$. If $n \geq i$ and $n \geq j$, then $n|g_i(y)| \geq j|g_j(y)| > 1$ on a neighborhood of $x$. Thus $f_n$ vanishes on that neighborhood provided $n \geq \max\{i, j\}$. This shows that $\{f_n\}$ is locally finite. Therefore $f = 1 + \sum f_n$ is an element of $C(X)$. On $F_n$, we have $g_i(x) = 0$ and $f_i(x) = 1$ for $i \leq n$. Thus $f(x) \geq n + 1 > |h(x)|$ on $F_n - F_{n+1}$. This proves that $f \geq |h|$.

**Remark 1.** It should be noted that the zero-sets play the same role in the above characterization that the closed $G_\delta$-sets play in Dowker’s characterization [1] of normal and countably paracompact spaces. This is not surprising since in normal spaces the zero-sets are precisely the closed $G_\delta$-sets.

**Remark 2.** In a completely regular space, the cozero-sets form a base for the topology. Hence, in such spaces, every cover has a cozero refinement. This fact emphasizes the significance of the local finiteness and countability requirements in (f) and (h).

**Remark 3.** If the word “increasing” is deleted from statements (d) through (h), and “decreasing” is deleted from (i), then each becomes a characterization of normal and countably paracompact spaces. [6, 1.1 and 1.2] or [7].

We are now able to state the following corollaries. Corollary 2 was originally proved by Horne [2].

**Corollary 2.** (i) A cb-space is countably paracompact.
(ii) A normal and countably paracompact space is a cb-space.

**Corollary 3.** A countably compact space is a cb-space.

**Corollary 4.** A closed subspace of a cb-space is a cb-space.

A subset $A$ of a topological space $X$ is called a generalized cozero-set (generalized $F_\sigma$-set) if each open set containing $A$ contains a cozero-set (respectively, $F_\sigma$-set) containing $A$.

**Lemma 5.** In a normal space, a set is a generalized cozero-set if and only if it is a generalized $F_\sigma$-set.

**Proof.** Since a cozero-set is an $F_\sigma$-set, the “only if” part is trivial. To prove the “if” part, it suffices to show that in a normal space each $F_\sigma$-set is a generalized cozero-set. Let $A = \bigcup F_n$ where each $F_n$ is closed. Given an open set $G \supseteq A$, there exists a sequence $\{Z_n\}$ of zero-sets such that $F_n \subseteq X - Z_n \subseteq G$. If $Z = \bigcap Z_n$, then $Z$ is a zero-set and $A \subseteq X - Z \subseteq G$. Thus $A$ is a generalized cozero-set.

**Theorem 6.** Each generalized cozero-subspace of a cb-space is a cb-space.
Proof. First, we shall consider a special case. Let $X$ be a $cb$-space and $Y$ be a cozero-set in $X$. Then there exists $g \in C(X)$, $0 \leq g$ such that $Y = X - Z(g)$. Set $F_n = \{ x : n^{-1} \leq g(x) \}$. Given an upper-semicontinuous function $h$ defined on $Y$, set $h_n(x) = h^+(x)$ on $F_n$ and $h_n(x) = 0$ on $X - F_n$. Then each $h_n$ is an upper-semicontinuous function on $X$; whence there exist $f_n \in C(X)$ such that $h_n \leq f_n$. Also, for $n > 2$ there exist $g_n \in C(X)$, $g_n \geq 0$ such that $Z(g_n) \supseteq F_{n-2}$ and $g_n(x) = 1$ on $X - F_{n-1}$ (appropriate modifications of $g$ will give such $g_n$). Set $g_1 = g_2 = 1$ and define $f = V_n(f_n g_n)$. Since $\{ g_n \}$ is a locally finite family in $C(Y)$, $f$ is an element of $C(Y)$. On $F_n - F_{n-1}$, $g_n(x) = 1$ and $h^+(x) = h_n(x) \leq f_n(x) \leq f(x)$. Therefore $h \leq f$. This proves the special case. The theorem follows by means of part (f) of Theorem 1, from the special case just proved.

The following previously known ([5] or [6, Corollary 1.6]) result is an immediate consequence of Lemma 5 and Theorem 6.

Corollary 7. Each generalized $F_\sigma$-subspace of a normal and countably paracompact space is normal and countably paracompact.

As noted in [2], a product of $cb$-spaces may fail to be cb or even countably paracompact. Also, the product of a normal and countably compact space with a compact space may fail to be normal [3, 8M.4]. The following theorem displays a property of $cb$-spaces which is not possessed by normal and countably paracompact spaces.

Theorem 8. The product of a $cb$-space and a locally compact, paracompact (Hausdorff) space is a $cb$-space.

Proof. First, consider the special case where $Y$ is compact. Let $\{ F_n \}$ be a decreasing sequence of closed sets in $X \times Y$ with empty intersection and let $A_n$ be the projection of $F_n$ into $X$. Since $Y$ is compact, the projection of $X \times Y$ onto $X$ is a closed mapping. Hence $\{ A_n \}$ is a decreasing sequence of closed sets in $X$. Since, for each $x \in X$, the set $\{ x \} \times Y$ is compact, this set fails to meet some $F_n$. Therefore $\{ A_n \}$ has empty intersection. According to Theorem 1, there exists a sequence $\{ Z_n \}$ of zero-sets in $X$ with empty intersection such that $Z_n \supseteq A_n$. Then $\{ Z_n \times Y \}$ is a sequence of zero-sets in $X \times Y$ with empty intersection such that $Z_n \times Y \supseteq F_n$. This proves the special case. Next, let $Y$ be locally compact and paracompact. Then there exists a locally finite cover $U$ of $Y$ consisting of relatively compact sets. Since $Y$ is normal, there exists a mapping $F$ from $U$ to the topology on $Y$ such that $\text{cl} F(U) \subseteq U$ and $\{ F(U) : U \in U \}$ covers $Y$. Let $h$ be a locally bounded function on $X \times Y$. By the special case just proved $X \times \text{cl} U$ is a $cb$-space. Thus, since $X \times F(U)$ and
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\[ X \times (Y - U) \] are completely separated, it follows that there exists \( f_U \subseteq C(X \times Y) \) for which \( f_U(x, y) \geq |h(x, y)| \) on \( X \times F(U) \) while \( f_U \) vanishes outside \( X \times U \). Since \( \mathcal{U} \) is locally finite, \( f = \bigvee_{U \in \mathcal{U}} f_U \) exists in \( C(X \times Y) \). Clearly, \( f \geq |h| \). This proves that \( X \times Y \) is a cb-space.

We shall now show that for certain non-normal spaces the cb-property is equivalent to countable paracompactness. A topological space \( X \) is pseudocompact if every real valued continuous function on \( X \) is bounded.

**Theorem 9.** Let \( X \) be a completely regular pseudocompact space. Then the following statements are equivalent:

(i) \( X \) is countably compact.

(ii) \( X \) is a cb-space.

(iii) \( X \) is countably paracompact.

**Proof.** The implications (i) \( \rightarrow \) (ii) and (ii) \( \rightarrow \) (iii) follow from Corollaries 2 and 3. We shall prove that (iii) \( \rightarrow \) (i). Suppose \( X \) is countably paracompact but not countably compact. Then there is a countably infinite set \( F = \{ x_1, x_2, \ldots, x_n, \ldots \} \) which has no limit points. For each \( n \), let \( U_n \) be an open set such that \( F \cap U_n = \{ x_n \} \). Then \( \{ U_n \} \cup \{ X - F \} \) is a cover of \( X \). This cover has a locally finite refinement \( \{ V_n \} \cup \{ X - F \} \) such that \( V_n \subset U_n \). Then \( x_n \in V_n \). Let \( f_n \in C(X) \) be such that \( f_n \) vanishes on \( X - V_n \) while \( f_n(x_n) = n \). Since \( \{ V_n \} \) is locally finite, \( f = \bigvee f_n \) exists in \( C(X) \). Clearly, \( f \) is unbounded. This is a contradiction since \( X \) is pseudocompact.

**Remark.** Theorem 9 generalizes to non-normal spaces the part of Theorem 1.8 in [6] for which \( m = \aleph_0 \).

**Theorem 10.** Let \( X \) be a topological space. Then the following statements are equivalent.

(i) \( X \) is countably paracompact.

(ii) For each increasing cover \( \{ U_n \} \) of \( X \), there exists a refinement \( \{ V_n \} \) such that \( \text{cl} V_n \subset U_n \).

(iii) For each locally bounded function \( h \) defined on \( X \), there exists a locally bounded lower-semicontinuous function \( g \) such that \( |h| \leq g \).

**Proof.** The equivalence of (i) and the closed set dual of (ii) is proved in [4].

(ii) \( \rightarrow \) (iii). Let \( h \) be locally bounded and set

\[ U_n = \text{int} \{ x : \ |h(x)| \leq n \}. \]

Then \( \{ U_n \} \) is an increasing cover for \( X \). Let \( \{ V_n \} \) be a refinement such that \( \text{cl} V_n \subset U_n \) and define \( g(x) = \sup \{ n : x \in U_k, \text{cl} V_k \} \).

Then \( g \) is lower-semicontinuous and \( g(x) \leq n \) on \( V_n \). Thus \( g \) is locally
bounded. If $g(x) = n$, then $x \in \text{cl} V_n \subset U_n$; whence $|h(x)| \leq n$. Therefore $|h| \leq g$.

(iii)$\rightarrow$(i). It suffices to show that (iii) implies that each increasing cover of $X$ has a locally finite refinement. Let $\{U_n\}$ be an increasing cover of $X$ and define $h(x) = \inf \{n: x \in U_n\}$. Then $h$ is locally bounded since it is positive and upper-semicontinuous. Let $g$ be a locally bounded, lower-semicontinuous function such that $g \geq h$. Set $V_n = \{x: n - 1 < g(x)\} \cap \{x: h(x) < n + 1\}$. Then $\{V_n\}$ is a refinement of $\{U_n\}$. Also, $\{V_n\}$ is locally finite since $g$ is locally bounded.

A topological space is extremally disconnected if every open set has an open closure.

**Theorem 11.** Let $X$ be an extremally disconnected space. Then $X$ is a cb-space if and only if it is countably paracompact.

**Proof.** We need prove only the "if" part. Suppose $X$ is countably paracompact and let $\{U_n\}$ be an increasing cover of $X$. Then there exists a refinement $\{V_n\}$ such that $\text{cl} V_n \subset U_n$. Since $X$ is extremally disconnected, $\text{cl} V_n$ is open. Therefore $\{\text{cl} V_n\}$ is a cozero refinement of $\{U_n\}$. That $X$ is a cb-space, now follows from Theorem 1.

*Added in proof.* A locally compact, countably paracompact space need not be a cb-space. See the example at the end of §3 in [8].

**References**


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