SOME PROPERTIES OF CERTAIN SETS OF COPRIME INTEGERS

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1. Introduction. The set \( P(n) \) of all primes less than or equal to \( n \) has the obvious property that it contains exactly one multiple of each prime less than or equal to \( n \). We use this partial description of \( P(n) \) as a basis for the following

Definition 1.1. An increasing sequence \( \{a_1, \ldots, a_k\} \) of integers greater than 1 is a coprime chain iff it contains exactly one multiple of each prime equal to or less than \( a_k \).

The following is a list of all coprime chains \( \{a_1, \ldots, a_k\} \) with \( a_k \leq 13 \):

\[
\begin{align*}
\{2\}; \\
2,3; \\
3,4; \\
2,3,5, \{3,4,5\}; \\
5,6; \\
2,3,5,7, \{3,4,5,7\}, \{5,6,7\}; \\
3,5,7,8; \\
2,5,7,9, \{4,5,7,9\}, \{5,7,8,9\}; \\
3,7,10; \\
2,3,5,7,11, \{3,4,5,7,11\}, \{3,5,7,8,11\}, \{2,5,7,9,11\}, \{4,5,7,9,11\}; \\
5,7,8,9,11, \{5,6,7,11\}, \{3,7,10,11\}, \{7,9,10,11\}; \\
5,7,11,12; \\
2,3,5,7,11,13, \{3,4,5,7,11,13\}, \{3,5,7,8,11,13\}, \{2,5,7,9,11,13\}; \\
4,5,7,9,11,13, \{5,7,8,9,11,13\}, \{5,6,7,11,13\}, \{3,7,10,11,13\}; \\
7,9,10,11,13, \{5,7,11,12,13\}.
\end{align*}
\]

The notation \( A(n), B(n), \) etc., will be used to designate coprime chains having \( n \) as largest member.

In this paper we are mainly concerned with finding functions asymptotic to the sum of the \( r \)th powers of the members of a coprime chain \( A(n) \). A later paper will deal with the number of coprime chains with largest member \( n \).

All assumed results are well known and can be found in [1].

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2. **The sum of powers of members of a coprime chain.** The brief table of coprime chains given in the introduction suggests that a coprime chain \( A(n) \) has, asymptotically, \( \pi(n) \) members. We will prove a stronger theorem but we first require the

**Lemma 2.1.**

\[
\sum_{p \leq n} p^r \sim \frac{n^{r+1}}{(r+1) \log n}, \quad r > -1.
\]

**Proof.** We note that for \( r = 0 \) this result is just the prime number theorem. For \( r > -1 \) partial summation gives

\[
\sum_{p \leq n} p^r = \pi(n)n^r - r \int_2^n \pi(t)t^{r-1} dt.
\]

But, since

\[
\int_2^n \pi(t)t^{r-1} dt \sim \int_2^n \left( \frac{1}{\log t} - \frac{1}{(r+1) \log^2 t} \right) dt \sim \frac{n^{r+1}}{(r+1) \log n},
\]

we have the desired result.

**Theorem 2.2.**

\[
\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r + O\left( \frac{n^{(c+1)/2}}{\log n} \right),
\]

where \( c = 1 \) for \( -1 < r \leq 0 \) and \( c = 2 \) for \( r > 0 \).

**Proof.** Partition \( A(n) \) into the sets \( P = \{ p \in A(n) \mid p \text{ is prime} \} \) and \( M = A(n) - P \). Then

\[
\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r - \sum_{p \leq n; p \notin P} p^r + \sum_{m \in M} m^r.
\]

Each member of \( M \) is divisible by some prime less than or equal to \( \sqrt{n} \) and, since the members of \( M \) are coprime in pairs, \( M \) has at most \( \pi(\sqrt{n}) \) members.

(i) Assume \( -1 < r \leq 0 \). For each \( m \) in \( M \) choose a prime divisor \( q \) of \( m \). Then

\[
\sum_{m \in M} m^r \leq \sum_{m \in M} q^r \leq \sum_{p \leq \sqrt{n}} p^r = O\left( \frac{n^{(r+1)/2}}{\log n} \right)
\]

by Lemma 2.1.
To complete the proof of this part of the theorem we note that there are at most $\pi(\sqrt{n})$ primes $p$ satisfying $\sqrt{n} < p \leq n$, $p \in \mathbb{P}$. Thus

$$\sum_{p \leq \sqrt{n}} p^r \leq \sum_{p \leq \sqrt{n}} p^r + \sum_{\sqrt{n} < p \leq n} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right).$$

(ii) Assume $r > 0$. Choose $n_0 > 2^{2/r}$. Then, for $n > n_0$,

$$\sum_{p \leq \sqrt{n}} p^r - \sum_{p > \sqrt{n}} p^r = \sum_{\sqrt{n} < p \leq n} p^r \leq \sum_{a \in A(n)} a^r \leq \sum_{p \leq \sqrt{n}} p^r + \sum_{m \in M} m^r.$$

Now

$$\sum_{m \in M} m^r \leq n^r \pi(\sqrt{n}) = O\left(\frac{n^{(r+1)/2}}{\log n}\right)$$

and

$$\sum_{p \leq \sqrt{n}} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right), \quad \text{which gives} \quad \sum_{p > \sqrt{n}} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right).$$

This completes the proof of the theorem.

Applying Lemma 2.1 to the above result we obtain the Corollary 2.3.

$$\sum_{a \in A(n)} a^r \sim \frac{n^{r+1}}{(r + 1) \log n}, \quad r > -1.$$

As the next theorem shows, Theorem 2.2 is the best possible, in that no error term of lower order will suffice.

**Theorem 2.4.** For all sufficiently large $n$, there exist coprime chains $A(n)$ and $B(n)$ such that

$$\left| \sum_{a \in A(n)} a^r - \sum_{b \in B(n)} b^r \right| \leq c_1 \frac{n^{(r+1)/2}}{\log n},$$

where $c$ is defined as in Theorem 2.2 and $c_1$ is a constant depending on $r$ only and is positive for $-1 < r \neq 0$.

**Proof.** Let $\{a_1, \cdots, a_{k-1}\}$ be the set of primes less than $n$ and not dividing $n$. Let $a_i = q_i \log n$ for $i = 1, \cdots, k-1$, $a_k = n$. Then $A(n) = \{a_1, \cdots, a_k\}$ and $B = \{b_1, \cdots, b_k\}$ are both coprime chains.

(i) Assume $-1 < r < 0$. Then
Theorem 2.5. If, for each $n$, coprime chains $A(n)$ and $B(n)$ are chosen so that $\sum_{a \in A(n)} 1/a$ and $\sum_{b \in B(n)} 1/b$ are maximal and minimal, respectively, then

$$\sum_{a \in A(n)} \frac{1}{a} \sim \log \log n \quad \text{and} \quad \sum_{b \in B(n)} \frac{1}{b} \to \log 2 \quad \text{as} \quad n \to \infty.$$
\[
\sum_{a \in A(n)} \frac{1}{a} = \sum_{p \mid n} \frac{1}{p} - \sum_{p \mid n \in P} \frac{1}{p} + \frac{1}{n} = \log \log n - \sum_{p \mid n} \frac{1}{p} + O(1).
\]

Since \(n\) has no more than \(2 \log n\) distinct prime divisors, it follows that
\[
\sum_{p \mid n} \frac{1}{p} = O(\log \log \log n),
\]
for all sufficiently large \(n\). Thus
\[
\sum_{a \in A(n)} \frac{1}{a} \sim \log \log n.
\]

To complete the proof we now consider any coprime chain \(B(n)\) chosen so that \(\sum_{b \in B(n)} 1/b\) is minimal for fixed \(n\) and note that \(B(n)\) can contain no number less than or equal to \(\sqrt{n}\). We define \(P\) and \(M\) as in the proof of Theorem 2.2. Then
\[
\sum_{b \in B(n)} \frac{1}{b} = \sum_{p \in P} \frac{1}{p} + \sum_{m \in M} \frac{1}{m}
\]
\[
= \sum_{\sqrt{n} < p \mid n; p \in P} \frac{1}{p} - \sum_{\sqrt{n} < p \mid n; p \in P} \frac{1}{p} + O\left(\sum_{m \in M} \frac{1}{\sqrt{n}}\right)
\]
\[
= \log 2 - \sum_{\sqrt{n} < p \mid n; p \in P} \frac{1}{p} + O\left(\frac{\pi(\sqrt{n})}{\sqrt{n}}\right)
\]
\[
= \log 2 - \sum_{\sqrt{n} < p \mid n; p \in P} \frac{1}{p} + o(1).
\]

Again, using the fact that \(M\) has at most \(\pi(\sqrt{n})\) members, we have
\[
\sum_{\sqrt{n} < p \mid n; p \in P} \frac{1}{p} = O\left(\frac{\pi(\sqrt{n})}{\sqrt{n}}\right) = o(1).
\]
Hence
\[
\sum_{b \in B(n)} \frac{1}{b} \to \log 2 \quad \text{as} \quad n \to \infty,
\]
and the proof of the theorem is complete.

If \(\{A(n)\}\) is any sequence of coprime chains, then the sequence whose members are \(\sum_{a \in A(n)} a^r\) is bounded for \(r < -1\), but for certain sequences \(\{B(n)\}\) we may obtain a more precise result.

**Theorem 2.6.** There exists a sequence \(\{B(n)\}\) of coprime chains
such that the sequence whose members are \( \sum_{b \in B(n)} b^r \) converges to 0 for all \( r < -1 \).

**Proof.** Assume \( r < -1 \), \( 0 < \varepsilon < 1 \) given. Choose \( n_0 \) so that \( n_0^{(r+1)/2} < \varepsilon / \log 2 \). Let \( \{ B(n) \} \) be a sequence of coprime chains chosen so that, for fixed \( n \), \( \sum_{b \in B(n)} 1/b \) is minimal. By Theorem 2.5 there is an \( n_1 \) so that \( \sum_{b \in B(n_1)} 1/b < \varepsilon + \log 2 \) for all \( n > n_1 \). Since \( B(n) \) contains no number less than or equal to \( \sqrt{n} \) we have \( b^{1+r} < n^{(r+1)/2} \) for all \( b \) in \( B(n) \). Thus, for all \( n > n_0 n_1 \), we have \( b^r < (1/b) \varepsilon / \log 2 \) for each \( b \) in \( B(n) \) and, hence,

\[
0 < \sum_{b \in B(n_1)} b^r < \frac{\varepsilon}{\log 2} \sum_{b \in B(n_1)} \frac{1}{b} < \varepsilon + \frac{\varepsilon^2}{\log 2} < 3 \varepsilon,
\]

and the proof is complete.

3. Coprime chains of maximal and minimal length.

**Definition 3.1.** For each \( n > 1 \) choose coprime chains \( A(n) \) and \( B(n) \) so that \( \sum_{a \in A(n)} 1 \) and \( \sum_{b \in B(n)} 1 \) are maximal and minimal, respectively. Define

\[
m(n) = \sum_{a \in A(n)} 1 \quad \text{and} \quad l(n) = \sum_{b \in B(n)} 1.
\]

Now \( m(n) \) and \( l(n) \) are about the same size; more precisely, setting \( r = 0 \) in Corollary 2.3 gives \( m(n) \sim l(n) \sim n / \log n \). However, we can make more precise statements about both \( m(n) \) and \( l(n) \).

**Theorem 3.2.** \( l(n) \) assumes every positive integral value.

**Proof.** From the table in the first section, \( l(2) = 1 \) and, since \( l(n) \to \infty \), it suffices to show \( l(n+1) \leq l(n) + 1 \), \( n > 1 \).

Let \( \{ a_1, \ldots, a_k = n \} \) be a coprime chain of minimal length with \( a_i, i = 1, \ldots, k-1 \), square-free. Let \( b_i = a_i/(a_i, n+1), i = 1, \ldots, k-1 \). Then the members of \( B(n+1) = \{ b_1, \ldots, b_{k-1}, n, n+1 \} \) are relatively prime in pairs and every prime less than or equal to \( n+1 \) divides some member of \( B(n+1) \). Thus, if all 1's are deleted from \( B(n+1) \) and the remaining members are properly reordered, we obtain a coprime chain. Then \( l(n+1) \leq k+1 = l(n) + 1 \) and the proof may be completed by induction.

**Theorem 3.3.** \( m(n) = \pi(n) - \omega(n) + 1 \), where \( \omega(n) \) is the number of different prime factors of \( n \).

**Proof.** Clearly a coprime chain of maximal length can be constructed by using only \( n \) and all primes less than and relatively prime to \( n \).
Corollary 3.4. \( m(n) \) assumes every positive integral value.

Proof. Letting \( n = p_k \) in the previous theorem we obtain \( m(p_k) = k \).

In Theorem 2.4 the restriction \( r \neq 0 \) is unnecessary in view of the following

Theorem 3.5. Given \( \epsilon > 0 \), \( m(n) - l(n) > (1 - \epsilon) \sqrt{n}/\log n \) for all sufficiently large \( n \).

Proof. If \( \{a_1, a_2, \ldots, a_{2k-1}, a_{2k}, a_{2k+1}, \ldots, n\} \) is a coprime chain and \( a_{2k} < \sqrt{n} \), then \( \{a_1a_2, a_3a_4, \ldots, a_{2k-1}a_{2k}, a_{2k+1}, \ldots, n\} \) can be reordered to form a coprime chain. Now the coprime chain containing \( n \) as largest member and all primes less than and relatively prime to \( n \) contains at least \( \pi(\sqrt{n}) - \omega(n) - 1 \) members less than \( \sqrt{n} \). By pairing these members as indicated above we can form a coprime chain with at most \( m(n) - \frac{\pi(\sqrt{n})}{2} - \omega(n) - 1 \) members. But since \( \omega(n) < 2 \log n \) we have

\[
l(n) \leq m(n) - \frac{1}{2} \left[ \pi(\sqrt{n}) - 2 \log n - 1 \right] = m(n) - \frac{\sqrt{n}}{\log n} \{1 + o(1)\}.
\]

Reference