1. Introduction. The set $P(n)$ of all primes less than or equal to $n$ has the obvious property that it contains exactly one multiple of each prime less than or equal to $n$. We use this partial description of $P(n)$ as a basis for the following.

**Definition 1.1.** An increasing sequence $\{a_1, \ldots, a_k\}$ of integers greater than 1 is a coprime chain iff it contains exactly one multiple of each prime equal to or less than $a_k$.

The following is a list of all coprime chains $\{a_1, \ldots, a_k\}$ with $a_k \leq 13$.

- \{2\};
- \{2,3\};
- \{3,4\};
- \{2,3,5\}, \{3,4,5\};
- \{5,6\};
- \{2,3,5,7\}, \{3,4,5,7\}, \{5,6,7\};
- \{3,5,7,8\};
- \{2,5,7,9\}, \{4,5,7,9\}, \{5,7,8,9\};
- \{3,7,10\}, \{7,9,10\};
- \{2,3,5,7,11\}, \{3,4,5,7,11\}, \{3,5,7,8,11\}, \{2,5,7,9,11\}, \{4,5,7,9,11\};
- \{5,7,8,9,11\}, \{5,6,7,11\}, \{3,7,10,11\}, \{7,9,10,11\};
- \{5,7,11,12\};
- \{2,3,5,7,11,13\}, \{3,4,5,7,11,13\}, \{3,5,7,8,11,13\}, \{2,5,7,9,11,13\};
- \{4,5,7,9,11,13\}, \{5,7,8,9,11,13\}, \{5,6,7,11,13\}, \{3,7,10,11,13\};
- \{7,9,10,11,13\}, \{5,7,11,12,13\}.

The notation $A(n)$, $B(n)$, etc., will be used to designate coprime chains having $n$ as largest member.

In this paper we are mainly concerned with finding functions asymptotic to the sum of the $r$th powers of the members of a coprime chain $A(n)$. A later paper will deal with the number of coprime chains with largest member $n$.

All assumed results are well known and can be found in [1].
2. The sum of powers of members of a coprime chain. The brief table of coprime chains given in the introduction suggests that a coprime chain \( A(n) \) has, asymptotically, \( \pi(n) \) members. We will prove a stronger theorem but we first require the

**Lemma 2.1.**

\[
\sum_{p \leq n} p^r \sim \frac{n^{r+1}}{(r + 1) \log n}, \quad r > -1.
\]

**Proof.** We note that for \( r = 0 \) this result is just the prime number theorem. For \( r > -1 \) partial summation gives

\[
\sum_{p \leq n} p^r = \pi(n)n^r - r \int_2^n \pi(t)t^{r-1} dt.
\]

But, since

\[
\int_2^n \pi(t)t^{r-1} dt \sim \int_2^n \left( \frac{1}{\log t} - \frac{1}{(r+1) \log^2 t} \right) dt \sim \frac{n^{r+1}}{(r + 1) \log n},
\]

we have the desired result.

**Theorem 2.2.**

\[
\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r + O\left( \frac{n^{(c+1)/2}}{\log n} \right),
\]

where \( c = 1 \) for \( -1 < r \leq 0 \) and \( c = 2 \) for \( r > 0 \).

**Proof.** Partition \( A(n) \) into the sets \( P = \{ p \in A(n) \mid p \text{ is prime} \} \) and \( M = A(n) - P \). Then

\[
\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r - \sum_{p \in P, \; p \leq \sqrt{n}} p^r + \sum_{m \in M} m^r.
\]

Each member of \( M \) is divisible by some prime less than or equal to \( \sqrt{n} \) and, since the members of \( M \) are coprime in pairs, \( M \) has at most \( \pi(\sqrt{n}) \) members.

(i) Assume \(-1 < r \leq 0\). For each \( m \) in \( M \) choose a prime divisor \( q \) of \( m \). Then

\[
\sum_{m \in M} m^r \leq \sum_{m \in M} q^r \leq \sum_{p \leq \sqrt{n}} p^r = O\left( \frac{n^{(r+1)/2}}{\log n} \right)
\]

by Lemma 2.1.
To complete the proof of this part of the theorem we note that there are at most \( \pi(\sqrt{n}) \) primes \( p \) satisfying \( \sqrt{n} < p \leq n \). Thus

\[
\sum_{p \leq n; p \in \mathbb{P}} p^r \leq \sum_{p \leq \sqrt{n}} p^r + \sum_{\sqrt{n} < p \leq n; p \in \mathbb{P}} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right).
\]

(ii) Assume \( r > 0 \). Choose \( n_0 > 2^{2/r} \). Then, for \( n > n_0 \),

\[
\sum_{p \leq n} p^r - \sum_{p \leq n} p^r = \sum_{\sqrt{n} < p \leq n} \frac{p^r}{(r+1) \log n}, \quad a^r \leq \sum_{p \leq n} p^r + \sum_{m \in M} m^r.
\]

Now

\[
\sum_{m \in M} m^r \leq n^r \pi(\sqrt{n}) = O\left(\frac{n^{(r+1)/2}}{\log n}\right)
\]

and

\[
\sum_{p \leq n} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right), \quad \text{which gives} \quad \sum_{p \leq n} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right).
\]

This completes the proof of the theorem.

Applying Lemma 2.1 to the above result we obtain the

**Corollary 2.3.**

\[
\sum_{a \in A(n)} a^r \sim \frac{n^{r+1}}{(r+1) \log n}, \quad r > -1.
\]

As the next theorem shows, Theorem 2.2 is the best possible, in that no error term of lower order will suffice.

**Theorem 2.4.** For all sufficiently large \( n \), there exist coprime chains \( A(n) \) and \( B(n) \) such that

\[
\left| \sum_{a \in A(n)} a^r - \sum_{b \in B(n)} b^r \right| \leq \frac{c_1 n^{(r+1)/2}}{\log n},
\]

where \( c \) is defined as in Theorem 2.2 and \( c_1 \) is a constant depending on \( r \) only and is positive for \(-1 < r \neq 0\).

**Proof.** Let \( \{q_1, \ldots, q_{k-1}\} \) be the set of primes less than \( n \) and not dividing \( n \). Let \( a_i = q_i \log n / \log q_i \), \( b_i = q_i \), \( i = 1, \ldots, k-1 \), \( a_k = b_k = n \). Then \( A(n) = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_k\} \) are both coprime chains.

(i) Assume \(-1 < r < 0\). Then
\[
\left| \sum_{a \in A(n)} a^r - \sum_{b \in B(n)} b^r \right|
= \sum_{p < n} p^r - \sum_{p \in n} p^r - \sum_{p < n} p^r[\log n/\log p] + \sum_{p | n} \frac{\log n}{\log p}
\]
\[
\geq \sum_{p \in \sqrt[n]{n}} p^r - \sum_{p \in n} p^r - \sum_{p \in \sqrt[n]{n}} p^r[\log n/\log p]
\]
\[
\geq \frac{1 + o(1)}{\log n} \frac{n^{(r+1)/2}}{r + 1} - \sum_{\sqrt[n]{n}} 1 - \sum_{p \in \sqrt[n]{n}} p^r[\log n/\log p - 1]
\]
\[
\geq \frac{2(1 + o(1))}{\log n} \frac{n^{(r+1)/2}}{r + 1} - \sum_{p \in \sqrt[n]{n}} n^r p^{-\frac{r}{2}} = \left\{1 + o(1)\right\} \frac{-4r}{1 - r^2} \frac{n^{(r+1)/2}}{\log n}
\]

(ii) Assume \( r > 0 \). Then
\[
\left| \sum_{a \in A(n)} a^r - \sum_{b \in B(n)} b^r \right|
\geq \sum_{p \in \sqrt[n]{n}} p^r[\log n/\log p] - \sum_{p \in n} p^r[\log n/\log p] - \sum_{p \in \sqrt[n]{n}} p^r
\]
\[
\geq \sum_{p \in \sqrt[n]{n}} p^r - \sum_{\sqrt[n]{n}} n^r - O\left(\frac{n^{(r+1)/2}}{\log n}\right)
\]
\[
\geq \left\{1 + o(1)\right\} \frac{n^{r+1/2}}{(r + \frac{1}{2}) \log n} + O(n^r \log n) + O(n^{(r+1)/2})
\]
\[
= \left\{1 + o(1)\right\} \frac{2}{2r + 1} \frac{n^{r+1/2}}{\log n}
\]

The above theorem is also valid for \( r = 0 \), as will be shown in Theorem 3.5.

The first major difference between coprime chains and sets of consecutive primes becomes apparent in the following

**Theorem 2.5.** If, for each \( n \), coprime chains \( A(n) \) and \( B(n) \) are chosen so that \( \sum_{a \in A(n)} 1/a \) and \( \sum_{b \in B(n)} 1/b \) are maximal and minimal, respectively, then

\[
\sum_{a \in A(n)} \frac{1}{a} \sim \log \log n \quad \text{and} \quad \sum_{b \in B(n)} \frac{1}{b} \to \log 2 \quad \text{as} \quad n \to \infty.
\]

**Proof.** Clearly \( A(n) - \{n\} \) is the set of primes less than \( n \) that do not divide \( n \). Hence
\[
\sum_{a \in A(n)} \frac{1}{a} = \sum_{p \mid n} \frac{1}{p} - \sum_{p \mid n} \frac{1}{p} + \frac{1}{n} = \log \log n - \sum_{p \mid n} \frac{1}{p} + O(1).
\]

Since \( n \) has no more than \( 2 \log n \) distinct prime divisors, it follows that

\[
\sum_{p \mid n} \frac{1}{p} = O(\log \log^2 n),
\]

for all sufficiently large \( n \). Thus

\[
\sum_{a \in A(n)} \frac{1}{a} \sim \log \log n.
\]

To complete the proof we now consider any coprime chain \( B(n) \) chosen so that \( \sum_{b \in B(n)} 1/b \) is minimal for fixed \( n \) and note that \( B(n) \) can contain no number less than or equal to \( \sqrt{n} \). We define \( P \) and \( M \) as in the proof of Theorem 2.2. Then

\[
\sum_{b \in B(n)} \frac{1}{b} = \sum_{p \in P} \frac{1}{p} + \sum_{m \in M} \frac{1}{m} = \sum_{\sqrt{n} \leq p \leq n} \frac{1}{p} + \sum_{\sqrt{n} < p \leq n} \frac{1}{p} + O\left( \sum_{m \in M} \frac{1}{\sqrt{n}} \right)
\]

\[
= \log 2 - \sum_{\sqrt{n} < p \leq n; p \in P} \frac{1}{p} + O\left( \frac{\pi(\sqrt{n})}{\sqrt{n}} \right)
\]

\[
= \log 2 - \sum_{\sqrt{n} < p \leq n; p \in P} \frac{1}{p} + o(1).
\]

Again, using the fact that \( M \) has at most \( \pi(\sqrt{n}) \) members, we have

\[
\sum_{\sqrt{n} < p \leq n; p \in P} \frac{1}{p} = O\left( \frac{\pi(\sqrt{n})}{\sqrt{n}} \right) = o(1).
\]

Hence

\[
\sum_{b \in B(n)} \frac{1}{b} \to \log 2 \quad \text{as} \quad n \to \infty,
\]

and the proof of the theorem is complete.

If \( \{A(n)\} \) is any sequence of coprime chains, then the sequence whose members are \( \sum_{a \in A(n)} a^r \) is bounded for \( r < -1 \), but for certain sequences \( \{B(n)\} \) we may obtain a more precise result.

**Theorem 2.6.** There exists a sequence \( \{B(n)\} \) of coprime chains
such that the sequence whose members are \( \sum_{b \in B(n)} b^r \) converges to 0 for all \( r < -1 \).

**Proof.** Assume \( r < -1 \), \( 0 < \epsilon < 1 \) given. Choose \( n_0 \) so that \( n_0^{(r+1)/2} < \epsilon/\log 2 \). Let \( \{ B(n) \} \) be a sequence of coprime chains chosen so that, for fixed \( n \), \( \sum_{b \in B(n)} 1/b \) is minimal. By Theorem 2.5 there is an \( n_1 \) so that \( \sum_{b \in B(n)} 1/b < \epsilon+\log 2 \) for all \( n > n_1 \). Since \( B(n) \) contains no number less than or equal to \( \sqrt{n} \) we have \( b^r < n^{(r+1)/2} \) for all \( b \) in \( B(n) \). Thus, for all \( n > n \circ n_1 \), we have \( b^r < (1/b)\epsilon/\log 2 \) for each \( b \) in \( B(n) \) and, hence,

\[
0 < \sum_{b \in B(n)} b^r < \frac{\epsilon}{\log 2} \sum_{b \in B(n)} \frac{1}{b} < \epsilon + \frac{\epsilon^2}{\log 2} < 3\epsilon,
\]

and the proof is complete.

3. Coprime chains of maximal and minimal length.

**Definition 3.1.** For each \( n > 1 \) choose coprime chains \( A(n) \) and \( B(n) \) so that \( \sum_{a \in A(n)} 1 \) and \( \sum_{b \in B(n)} 1 \) are maximal and minimal, respectively. Define

\[
m(n) = \sum_{a \in A(n)} 1 \quad \text{and} \quad l(n) = \sum_{b \in B(n)} 1.
\]

Now \( m(n) \) and \( l(n) \) are about the same size; more precisely, setting \( r = 0 \) in Corollary 2.3 gives \( m(n) \sim l(n) \sim n/\log n \). However, we can make more precise statements about both \( m(n) \) and \( l(n) \).

**Theorem 3.2.** \( l(n) \) assumes every positive integral value.

**Proof.** From the table in the first section, \( l(2) = 1 \) and, since \( l(n) \rightarrow \infty \), it suffices to show \( l(n+1) \leq l(n)+1 \), \( n > 1 \).

Let \( \{ a_i, \ldots, a_k = n \} \) be a coprime chain of minimal length with \( a_i, i = 1, \ldots, k-1 \), square-free. Let \( b_i = a_i/(a_i, n+1) \), \( i = 1, \ldots, k-1 \). Then the members of \( B(n+1) = \{ b_1, \ldots, b_{k-1}, n, n+1 \} \) are relatively prime in pairs and every prime less than or equal to \( n+1 \) divides some member of \( B(n+1) \). Thus, if all 1's are deleted from \( B(n+1) \) and the remaining members are properly reordered, we obtain a coprime chain. Then \( l(n+1) \leq k+1 = l(n)+1 \) and the proof may be completed by induction.

**Theorem 3.3.** \( m(n) = \pi(n) - \omega(n) + 1 \), where \( \omega(n) \) is the number of different prime factors of \( n \).

**Proof.** Clearly a coprime chain of maximal length can be constructed by using only \( n \) and all primes less than and relatively prime to \( n \).
Corollary 3.4. $m(n)$ assumes every positive integral value.

Proof. Letting $n = p_k$ in the previous theorem we obtain $m(p_k) = k$.
In Theorem 2.4 the restriction $r \neq 0$ is unnecessary in view of the following

Theorem 3.5. Given $\epsilon > 0$, $m(n) - l(n) > (1 - \epsilon)\sqrt{n}/\log n$ for all sufficiently large $n$.

Proof. If $\{a_1, a_2, \ldots, a_{2k-1}, a_{2k}, a_{2k+1}, \ldots, n\}$ is a coprime chain and $a_{2k} < \sqrt{n}$, then $\{a_1a_2, a_3a_4, \ldots, a_{2k-1}a_{2k}, a_{2k+1}, \ldots, n\}$ can be reordered to form a coprime chain. Now the coprime chain containing $n$ as largest member and all primes less than and relatively prime to $n$ contains at least $\pi(\sqrt{n}) - \omega(n) - 1$ members less than $\sqrt{n}$. By pairing these members as indicated above we can form a coprime chain with at most $m(n) - \frac{1}{2} [\pi(\sqrt{n}) - \omega(n) - 1]$ members. But since $\omega(n) < 2 \log n$ we have

$$l(n) \leq m(n) - \frac{1}{2} [\pi(\sqrt{n}) - 2 \log n - 1] = m(n) - \frac{\sqrt{n}}{\log n} \{1 + o(1)\}.$$ 

Reference


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