THE COHOMOLOGY RING OF A COMPACT LIE GROUP WITH BI-INARIANT METRIC

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I. Introduction. In this note we shall show that the adjoint operation * obtained from a bi-invariant riemannian metric on a compact Lie group induces an isomorphism between the cup and Pontrjagin products on the cohomology ring. This fact is easily and directly verifiable in the case of a torus, where, as we shall show elsewhere, it has interesting applications to the classical theory of abelian varieties; in fact, it motivates a definition of * on the numerical equivalence ring of an abstract polarized abelian variety.

II. Algebraic preliminaries. Let $E$ be an $n$-dimensional vector space over $R$, $\Lambda^p(E)$ the $p$-fold exterior product, and $\bar{E}$, $\Lambda^q(E)$, their respective dual spaces. There is a canonical isomorphism $i_p: \Lambda^p(E) \to \Lambda^p(\bar{E})$. An orientation of $E$ is an isomorphism $e: \Lambda^n(E) \cong R$. It gives rise to a dual orientation $\bar{e}: \Lambda^n(\bar{E}) \cong R$, and we denote the fundamental $n$-vector and $n$-covector by $e = e^{-1}(1)$ and $\bar{e} = \bar{e}^{-1}(1)$, respectively. One defines an isomorphism $j_p: \Lambda^p(E) \to \Lambda^{n-p}(E)$ by letting $j_p(\alpha)(\beta) = e(\alpha \wedge \beta)$. This gives an isomorphism $k_p = i_{n-p} \circ j_p: \Lambda^p(E) \cong \Lambda^{n-p}(E)$. For $\alpha \in \Lambda^p(E)$, $\beta \in \Lambda^q(E)$, let $\alpha \vee \beta = k^{-1}(k\alpha \wedge k\beta) \in \Lambda^{p+q}(E)$, and for $\beta \in \Lambda^q(E)$, $\alpha \wedge \beta = \alpha \wedge k^{-1}(\beta) \in \Lambda^{p+q}(E)$.

The composition map $T: E \oplus E \to E$, sending $\alpha \oplus \beta$ into $\alpha + \beta$, can be uniquely extended to an algebra homomorphism $\{T^p\}$, $T^p: \Lambda^p(E \oplus E) \to \Lambda^p(E)$. Since $\Lambda^p(E \oplus E) \cong \bigoplus_{r+s=p} \Lambda^r(E) \otimes \Lambda^s(E)$, we can define, for $\alpha \in \Lambda^r(E)$ and $\beta \in \Lambda^s(E)$, $T^{r+s}(\alpha \otimes \beta) \in \Lambda^{r+s}(E)$, and a simple computation shows this is equal to $\alpha \wedge \beta$. Also, $T: E \otimes E \to E$ may be dualized to give $\bar{T}: \bar{E} \otimes \bar{E} \to \bar{E}$, which extends to an algebra homomorphism $\{\bar{T}^p\}$, $\bar{T}^p: \Lambda^p(\bar{E} \otimes \bar{E}) \to \Lambda^p(E \oplus E)$. Given $\alpha \in \Lambda^p(E)$, $\beta \in \Lambda^q(E)$, we have $\alpha \otimes \beta \in \Lambda^{p+q}(E \oplus E)$ and $\alpha \wedge \beta \in \Lambda^{p+q}(E \oplus E)$. An easy computation shows that $T^{r+s}(\alpha \otimes \beta \otimes \beta') = \alpha \vee \beta$.

A quadratic form on $E$ is an isomorphism $\phi: E \to \bar{E}$, extendable uniquely to an algebra isomorphism $\{\phi^p\}$, $\phi^p: \Lambda^p(E) \to \Lambda^p(\bar{E})$. We define $*: \Lambda^p(E) \to \Lambda^{n-p}(E)$ by $*\alpha = k^{-1}\phi\alpha$. Note that $* (\alpha \wedge \beta) = k^{-1}(\phi\alpha \wedge \phi\beta) = k^{-1}(k^{-1}(k\alpha \wedge k\beta) = k^{-1}(k\alpha \wedge k\beta) = k^{-1}(k\alpha \wedge k\beta) = *\alpha \vee \beta$. So * is an isomorphism of the $\wedge$-algebra onto the $\vee$-algebra; if $\phi$ is unitary, i.e., $*1 = e$, the map is unit preserving.
Needless to say, the above considerations extend to the case where \( E \) is a vector bundle over a manifold \( M \); e.g., if \( E \) is the cotangent bundle of an oriented manifold, we recover the familiar \( * \) of Weitzenbock-Hodge.

III. Integration along the fibre. Let \( G \) be a compact oriented Lie group with dual Lie algebra \( E \) (whose orientation is \( e \)), and let \( \Phi : Y \to X \) be a principal fibre bundle with group \( G \), \( X \) and \( Y \) being compact and having orientations compatible with that of \( G \) (with respect to the local product structure). The orientation of \( E \) gives \( G \) a unique left-invariant Haar measure. For all \( x \in X \), there is a homeomorphism \( \psi_x : G \to \Phi^{-1}(x) \), sending the invariant \( n \)-vector \( e \) onto an invariant \( n \)-vector \( e_0 \) along the fibre. Let \( \alpha \) be a \( p \)-form on \( Y \); we define a \( (p - n) \)-form \( \Phi(\alpha) \) on \( X \) by integrating along the fibre. Precisely, \( \alpha \wedge e_0 \) along the fibre \( \Phi^{-1}(x) \) is a form annihilating all vectors tangent to the fibre; hence it may be integrated, by integrating its coefficients with respect to the Haar measure, to give a \( (p - n) \)-form \( \Phi(\alpha) \) at \( x \). The local product structure and the invariance of the measure guarantee that \( \Phi(\alpha) \) is well defined and differentiable, and it is easy to check that \( \Phi(\alpha) \) is closed if \( \alpha \) is.

Let \( y \) be a \( p \)-cohomology class in \( Y \), the de Rham class of a form \( \alpha \). Its Poincaré dual class \( p_y \in H^{k-p}(Y, \mathbb{R}) \) is defined as a linear functional on \( H^{k-p}(Y, \mathbb{R}) \) by \( p_y(\beta) = \int_Y \alpha \wedge \beta \), \( \beta \) being the de Rham class of a form \( \beta \). The map \( \Phi \) induces a map \( \Phi_* : H^{k-p}(Y, \mathbb{R}) \to H^{k-p}(X, \mathbb{R}) \) by letting \( \Phi_* (p_y)(w) = \int_X \alpha \wedge \Phi^* \gamma \), where \( w \) is the de Rham class of a \( (k-p) \)-form \( \gamma \) on \( X \). On the other hand, the Poincaré dual class of \( \Phi(\alpha) \) is defined as a functional on \( H^{p-q}(X, \mathbb{R}) \) by sending the de Rham class \( w \) of a form \( \gamma \) into \( \int_X \Phi(\alpha) \wedge \gamma \). A partition of unity and Fubini's theorem show immediately that \( P [ \Phi(\alpha) ] = \Phi_* (p_y) \), \( y \) again being the class of \( \alpha \). Hence \( \Phi \) induces on the de Rham cohomology the well-known Gysin homomorphism.

IV. The convolution product. Now let \( X = G \), \( Y = G \times G \) and \( \Phi \) be the composition map. Further, let \( \pi_1 \) and \( \pi_2 \) be the projections of \( G \times G \) onto its factors. If \( \alpha \) is a \( p \)-form, and \( \beta \) a \( q \)-form, on \( G \), the \( (p+q-n) \)-form \( \alpha \circ \beta = \Phi(\pi_1^* \alpha \wedge \pi_2^* \beta) \) will be called the convolution of \( \alpha \) and \( \beta \). (If \( p = q = n \), \( \alpha = fe \) and \( \beta = ge \), then \( \alpha \circ \beta = he \), and it is clear that the function \( h \) is the convolution, in the usual sense, of the functions \( f \) and \( g \).) From the last remarks in §III, it is clear that the convolution algebra on closed forms induces on the cohomology of \( G \) the algebra structure obtained by transposing (via Poincaré duality) the Pontrjagin algebra on homology to cohomology.

Now let \( \alpha \) be right invariant, and \( \beta \) left invariant. Then \( \pi_1^* \alpha \)
\( \wedge \pi_2 \pi_0 \) is invariant along the fibre \( \Phi^{-1}(1) \) so the value of \( \alpha \bullet \beta \) at 1 is merely the image under the canonical map \( \Lambda^{p+q} E \otimes E \rightarrow \Lambda^{p+q} E \) of the value of \( \pi_2 \alpha \wedge \pi_0 \beta \) at 1. (We are using the fact that the map \( \psi : G \rightarrow G \times G \) induces, on the dual Lie algebra, the canonical map of composition.) By the remarks in II, \( \alpha \bullet \beta \) evaluated at 1 is thus (the value of \( \alpha \) at 1) \( \vee \) (the value of \( \beta \) at 1).

Now assume that both \( \alpha \) and \( \beta \) are bi-invariant. Then \( \pi_1 \alpha \wedge \pi_2 \beta \) is bi-invariant on \( G \times G \); invariant, in particular, under the liftings via \( \Phi \) of both right and left translations on \( G \). Hence, \( \alpha \bullet \beta \) is bi-invariant, and so equals \( \alpha \vee \beta \).

V. Conclusion of the proof. We may map \( H^*(G, \mathbb{R}) \) into \( \Lambda^*(E) \) by sending each class into the unique bi-invariant form it contains, evaluated at 1. It is well known that this is a homomorphism of the cup-algebra into the \( \wedge \)-algebra, and we have shown above that it is also a homomorphism of the (transposed) Pontrjagin algebra into the \( \vee \)-algebra.

Now, let \( \phi \) be a bi-invariant riemannian metric on \( G \). (One is easily constructed by integrating a positive-definite quadratic form on \( E \) under the adjoint group, and then translating.) Then (e.g., since bi-invariant forms are harmonic) the action of \( \ast \) on \( \Lambda^*(E) \) is compatible with the action of \( \ast \) on cohomology. Letting \( \oplus \) denote the (cohomological) Pontrjagin product, we may combine this last remark with that at the end of §II to assert that, for any cohomology classes \( x \) and \( y \) on \( G \), \( \ast (x \cup y) = \ast x \oplus \ast y \).