HOMOTOPY FOR CELLULAR SET-VALUED FUNCTIONS

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1. Introduction. A. Granas asked the following question. If $F$ is an upper semi-continuous set-valued function on a compact metric space $M$ such that the image of each point of $M$ is a proper subcontinuum of $S^n$, then is $F$ "homotopic" to a single-valued continuous function? It was pointed out that care must be used in the definition of homotopy of set-valued functions, since the first natural candidate puts all upper semi-continuous set-valued functions into one class. In [3] and [2] studies were made of homotopies of set-valued functions subject to the restriction that $H(x, t)$ be acyclic (with respect to homology over $\mathbb{Z}_2$) for each $(x, t) \in M \times I$.

In this paper the homotopy problem is solved for those upper semi-continuous functions $F$ for which each $F(x)$ is a cellular subset of $S^n$. In particular, the class of cellular upper semi-continuous set-valued functions is partitioned into equivalence classes by the relation of cellular homotopy, each class contains single-valued continuous functions, and two single-valued continuous functions are in the same class if and only if they are homotopic in the usual sense.

A selection theorem which seems to be different from those discussed in the literature is proved in §2. It is shown that if $F$ is upper semi-continuous on $M$ and $F(x)$ is a cellular subset of $S^n$ for each $x \in M$, then there exists a continuous function $g: M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$. In addition to being the main tool used in the construction of the homotopies, this selection theorem is of interest in itself.

2. The selection theorem. A subset $A$ of $S^n$ is cellular if and only if there exists a sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of topological $n$-cells such that $A = \bigcap_{k=1}^{\infty} E_k$ and, for each $k$, $A \subseteq \text{interior } E_k$.

Let $M$ be an $m$-dimensional compact metric space and let $F$ be a set-valued function on $M$ such that:

(i) for each $x \in M$, $F(x)$ is a cellular subset of $S^n$, and
(ii) $F$ is upper semi-continuous.

A covering pair for $F$ and $M$ is an ordered pair $(G, D)$ such that:

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(i) \( G \) is a finite open covering of \( M \),
(ii) \( D \) is a function with domain \( G \) such that for each \( U \in G \), \( D(U) \) is a topological \( n \)-cell which is contained in \( S^n \), and
(iii) for each \( x \in M \), if \( x \in U \in G \) then \( F(x) \subset \text{interior} \ D(U) \).

**Lemma 1.** There exists a covering pair.

**Proof.** For each \( x \in M \), there exists a topological \( n \)-cell \( \Delta(x) \) such that \( F(x) \subset \text{interior} \ \Delta(x) \) and \( \Delta(x) \subset S^n \). For each \( x \in M \), there exists a neighborhood \( V(x) \) such that if \( t \in V(x) \) then \( F(t) \subset \text{interior} \ \Delta(x) \). \( \{ V(x) \mid x \in M \} \) is an open covering of \( M \), and, since \( M \) is compact, this covering has a finite subcovering \( G = \{ V(x_1), \ldots, V(x_k) \} \). We define \( D(V(x_j)) = \Delta(x_j) \) for \( j = 1, \ldots, k \). It is easy to verify that \((G, D)\) is a covering pair.

**Lemma 2.** If \((G, D)\) is a covering pair, then there exists a covering pair \((G^*, D^*)\) such that:

(i) \( G^* \) is a star refinement of \( G \), and
(ii) if \( U \in G \), \( U^* \in G^* \) and \( U^* \subset U \), then \( D^*(U^*) \subset D(U) \).

**Proof.** Let \( \lambda \) be the Lebesgue number of the covering \( G \). For each \( x \in M \), there is a topological \( n \)-cell \( \Delta(x) \) such that \( F(x) \subset \text{interior} \ \Delta(x) \) and such that if \( x \in U \in G \) then \( \Delta(x) \subset D(U) \). For each \( x \in M \), there is a neighborhood \( W(x) \) of \( x \) such that:

(i) \( W(x) \) is contained in the \( \lambda/3 \)-neighborhood of \( x \), and
(ii) if \( t \in W(x) \) then \( F(t) \subset \text{interior} \ \Delta(x) \). The set \( \{ W(x) \mid x \in M \} \) is an open covering of \( M \) and has a finite subcover \( G^* = \{ W(x_1), \ldots, W(x_k) \} \). We define \( D^*(W(x_j)) = \Delta(x_j) \) for \( j = 1, \ldots, k \). It is easy to verify that \((G^*, D^*)\) has the desired properties.

Let \( F \) be an upper semi-continuous set-valued function on a compact finite-dimensional metric space \( M \) such that, for each \( x \in M \), \( F(x) \) is a cellular subset of \( S^n \).

**Theorem 1.** There exists a single-valued continuous function \( g : M \to S^n \) such that \( g(x) \in S^n - F(x) \) for each \( x \in M \).

**Proof.** Let \( m \) be the dimension of \( M \). It follows from Lemmas 1 and 2 that there are covering pairs \((G_0, D_0)\), \ldots, \((G_{2m}, D_{2m})\) such that for each \( j, 1 \leq j \leq 2m \): (i) \( G_j \) is a star refinement of \( G_{j-1} \), and (ii) if \( U_j \in G_j \), \( U_{j-1} \in G_{j-1} \) and \( U_j \subset U_{j-1} \), then \( D_j(U_j) \subset D_{j-1}(U_{j-1}) \).

We choose a finite open covering \( G \) of \( M \) such that \( G \) is of order \( m \), \( G \) is a star refinement of \( G_{2m} \), and no proper subset of \( G \) covers \( M \). For each integer \( j, 0 \leq j \leq m \), we define \( K_j = \{ x \mid x \in M \text{ and } x \text{ is a member of at most } j+1 \text{ members of } G \} \). Each \( K_j \) is a closed subset of \( M \), and \( K_m = M \).
For each $V \in G$ and each integer $j$, $0 \leq j \leq 2m$, we select sets $\phi_j(V) \subseteq G_j$ such that $\text{St}(V) \subseteq \phi_{2m}(V)$ and $\text{St}(\phi_j(V)) \subseteq \phi_{j+1}(V)$, for $1 \leq j \leq 2m$. We define $\Phi_j(V) = D_j(\phi_j(V))$ for each $V \in G$ and $j = 0, \ldots, 2m$.

We are going to define (inductively) for each $j$, $0 \leq j \leq m$, a mapping $g_j: K_j \rightarrow S^n$ such that, for each $V \in G$, 

$$g_j(V \cap K_j) \subseteq \text{Cl}[S^n - \Phi_j(V)].$$

For each $V \in G$, we choose a point $p_V \in S^n - \Phi_0(V)$ and define $g_0(x) = p_V$ for each $x \in V \cap K_0$. Since $V \cap K_0$ is closed in $K_0$ for each $V \in G$, $g_0$ is continuous.

Now suppose $0 < j \leq m$ and $g_{j-1}$ has been defined. Let $\sigma = \{V_0, \ldots, V_j\}$ be a set of $j+1$ distinct members of $G$ such that $V_0 \cap \cdots \cap V_j \neq \emptyset$. We define $H_\sigma = (V_0 \cap \cdots \cap V_j) \cap K_j$ and $W_\sigma = K_j - \bigcup \{V \mid V \in G - \sigma\}$. Then $W_\sigma$ is closed in $K_j$ and $H_\sigma$ is open relative to $W_\sigma$. The mapping $g_{j-1}$ is defined on $W_\sigma - H_\sigma$ and $W_\sigma - H_\sigma$ is closed relative to $W_\sigma$. It is easy to see that

$$g_{j-1}[W_\sigma - H_\sigma] \subseteq \bigcup_{r=0}^j \text{Cl}[S^n - \Phi_{2j-2}(V_r)].$$

Since $\Phi_{2j-1}(V) \subseteq \text{St}(\Phi_{2j-1}(V_r)) \subseteq \Phi_{2j-2}(V_r)$ for $r = 0, \ldots, j$, $\Phi_{2j-1}(V_0) \subseteq \bigcap_{r=0}^j \Phi_{2j-2}(V_r)$. Therefore, $g_{j-1}[W_\sigma - H_\sigma] \subseteq \text{Cl}[S^n - \Phi_{2j-1}(V_0)]$.

The set $\text{Cl}[S^n - \Phi_{2j-1}(V_0)]$ is the union of a topological $(n-1)$-sphere $\Sigma$ and one of the components of $S^n - \Sigma$. It is known (see [1]) that such sets are absolute retracts. Since $\text{Cl}[S^n - \Phi_{2j-1}(V_0)]$ is an absolute retract, we can extend $g_{j-1}|(W_\sigma - H_\sigma)$ to a mapping 

$$\psi_\sigma: W_\sigma \rightarrow \text{Cl}[S^n - \Phi_{2j-1}(V_0)].$$

Since $V_r \subseteq \text{St}(V_0)$ for $r = 0, \ldots, j$, 

$$\Phi_{2j}(V_r) \subseteq \text{St}(\Phi_{2j}(V_0)) \subseteq \Phi_{2j-1}(V_0).$$

Thus $\Phi_{2j}(V_r) \subseteq \Phi_{2j-1}(V_0)$ and the range of $\psi_\sigma$ is contained in $\text{Cl}[[S^n - \Phi_{2j}(V_r)]$ for $r = 0, \ldots, j$. Thus $\psi_\sigma[V_r \cap W_\sigma] \subseteq \text{Cl}[S^n - \Phi_{2j}(V_r)]$ for $r = 0, \ldots, j$.

If $\sigma'$ is a different system of $j+1$ members of $G$ and $W_{\sigma'} \cap W_\sigma \neq \emptyset$, then $\psi_{\sigma'}(W_{\sigma'} \cap W_\sigma) = \psi_{\sigma'}(W_{\sigma'} \cap W_\sigma) = g_{j-1}|(W_{\sigma'} \cap W_\sigma)$. It follows that we can piece the mappings $\psi_\sigma$ and $g_{j-1}$ together to obtain a mapping $g_j: K_j \rightarrow S^n$. It is obvious that $g_j(V \cap K_j) \subseteq \text{Cl}[S^n - \Phi_j(V)]$ for each $V \in G$.

We define $g = g_m$. Since $K_m = M$, $g$ is a mapping of $M$ into $S^n$ such
that $g[V] \subset \text{Cl}[S^n - \Phi_{2m}(V)]$ for each $V \in G$. If $x \in V \in G$, then $x \in \Phi_{2m}(V)$ and, hence, $F(x) \subset \text{interior } \Phi_{2m}(V)$. It follows that if $x \in M$, then $g(x) \in S^n - F(x)$.

3. Homotopy for a class of set-valued functions. Let $M$ be a finite-dimensional compact metric space. We define $\Gamma(M, S^n)$ to be the set of all upper semi-continuous set-valued functions $F$ on $M$ such that for each $x \in M$, $F(x)$ is a cellular subset of $S^n$. We let $I = [0, 1]$.

Let $F$ and $G$ be members of $\Gamma(M, S^n)$. A function $H$ is a cellular homotopy relating $F$ to $G$ if:

(i) $H \in \Gamma(M \times I, S^n)$, and

(ii) for all $x \in M$, $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$.

If there exists a cellular homotopy relating $F$ to $G$, then we say that $F$ is homotopic to $G$ and write $F \sim G$. The relation $\sim$ is an equivalence relation and partitions $\Gamma(M, S^n)$ into equivalence classes which we call cellular homotopy classes.

Let $F \in \Gamma(M, S^n)$ and let $f : M \to S^n$ be a (single-valued) continuous function. A function $H$ is a special homotopy relating $F$ to $f$ if:

(i) $H$ is a cellular homotopy relating $F$ to $f$, and

(ii) for all $x \in M$ and $0 \leq t < 1$, $H(x, t)$ is homeomorphic to $F(x)$.

If $F$ is a single-valued function as well as $f$, then (ii) implies that $H$ is single-valued, and since upper semi-continuity is equivalent to continuity for single-valued functions, in this case $H$ is an ordinary homotopy.

Lemma 3. If $F \in \Gamma(M, S^n)$, then there exists a single-valued continuous function $f : M \to S^n$ and a special homotopy $H$ relating $F$ to $f$.

Proof. For each $p \in S^n$, we define a mapping $J_p : [S^n - p] \times I \to S^n$ by

$$J_p(x, t) = \frac{-tp + (1 - t)x}{\| -tp + (1 - t)x\|}$$

for $x \in S^n - p$, $0 \leq t \leq 1$. $J_p$ is a pseudo-isotopy, since the map $\phi_t$ defined by $\phi_t(x) = J_p(x, t)$ is a homeomorphism on $S^n - p$ if $0 \leq t < 1$, $\phi_0$ is the identity mapping on $S^n - p$, and $\phi_1$ is the constant map which takes $S^n - p$ into $-p$.

By Theorem 1, there is a mapping $g : M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$. We define $f(x) = -g(x)$ and

$$H(x, t) = \{ J_{\phi_t}(y, t) \mid y \in F(x) \}$$

for $x \in M$ and $0 \leq t \leq 1$. Obviously, $f$ is a continuous function on $M$ into $S^n$, and it is easy to verify that $H$ is a special homotopy relating $F$ to $f$. 

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Theorem 2. Each cellular homotopy class of $\Gamma(M, S^n)$ contains a single-valued continuous function $f: M \to S^n$.

Proof. This result follows immediately from Lemma 3 and the fact that special homotopies are cellular homotopies.

Our final theorem shows that the notion of cellular homotopy which we have defined for $\Gamma(M, S^n)$ is a true extension of the usual notion of homotopy for single-valued functions.

Theorem 3. If $f_0$ and $f_1$ are single-valued continuous functions on $M$ into $S^n$ and $H \in \Gamma(M \times I, S^n)$ is a cellular homotopy relating $f_0$ to $f_1$, then there exists a single-valued homotopy $h: M \times I \to S^n$ which relates $f_0$ to $f_1$ in the usual sense.

Proof. We apply Lemma 3 (replacing $M$ by $M \times I$ and $F$ by $H$) to obtain a single-valued continuous function $\phi: M \times I \to S^n$ and a special homotopy $K \in \Gamma((M \times I) \times I, S^n)$ relating $H$ to $\phi$. Now, for $(x, t) \in M \times I$, we define

$$h(x, t) = \begin{cases} 
K(x, 0, 3t) & \text{if } 0 \leq t \leq 1/3, \\
K(x, 3t - 1, 1) & \text{if } 1/3 \leq t \leq 2/3, \\
K(x, 1, 3 - 3t) & \text{if } 2/3 \leq t \leq 1.
\end{cases}$$

If $0 \leq t < 1/3$, then $h(x, t) = K(x, 0, 3t)$ is homeomorphic to $K(x, 0, 0) = H(x, 0) = f_0(x)$ and, hence, is a one-point set. Likewise, if $2/3 < t \leq 1$, then $h(x, t)$ is a one-point set. If $1/3 \leq t \leq 2/3$, then $h(x, t) = K(x, 3t - 1, 1) = (x, 3t - 1)$ and since $\phi$ is single-valued, $h(x, t)$ is a one-point set. Thus $h$ is a single-valued function on $M \times I$ into $S^n$.

Since $K$ is upper semi-continuous, $h$ is also upper semi-continuous. Since $h$ is single-valued, this implies that $h$ is continuous.

We have shown that $h: M \times I \to S^n$ is an ordinary single-valued homotopy. Since $h(x, 0) = K(x, 0, 0) = H(x, 0) = f_0(x)$ and $h(x, 1) = K(x, 1, 0) = H(x, 1) = f_1(x)$, $h$ relates $f_0$ to $f_1$ in the usual sense.

It should be remarked that it can be shown that the smallest equivalence relation containing both homotopies of single-valued functions and special homotopies is the relation generated by cellular homotopies. The proof is similar to that of Theorem 4.

Bibliography


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