HOMOTOPY FOR CELLULAR SET-VALUED FUNCTIONS

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1. Introduction. A. Granas asked the following question. If $F$ is an upper semi-continuous set-valued function on a compact metric space $M$ such that the image of each point of $M$ is a proper subcontinuum of $S^n$, then is $F$ "homotopic" to a single-valued continuous function? It was pointed out that care must be used in the definition of homotopy of set-valued functions, since the first natural candidate puts all upper semi-continuous set-valued functions into one class. In [3] and [2] studies were made of homotopies of set-valued functions subject to the restriction that $H(x, t)$ be acyclic (with respect to homology over $\mathbb{Z}_2$) for each $(x, t) \in M \times I$.

In this paper the homotopy problem is solved for those upper semi-continuous functions $F$ for which each $F(x)$ is a cellular subset of $S^n$. In particular, the class of cellular upper semi-continuous set-valued functions is partitioned into equivalence classes by the relation of cellular homotopy, each class contains single-valued continuous functions, and two single-valued continuous functions are in the same class if and only if they are homotopic in the usual sense.

A selection theorem which seems to be different from those discussed in the literature is proved in §2. It is shown that if $F$ is upper semi-continuous on $M$ and $F(x)$ is a cellular subset of $S^n$ for each $x \in M$, then there exists a continuous function $g: M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$. In addition to being the main tool used in the construction of the homotopies, this selection theorem is of interest in itself.

2. The selection theorem. A subset $A$ of $S^n$ is cellular if and only if there exists a sequence $E_1 \supset E_2 \supset E_3 \cdots$ of topological $n$-cells such that $A = \bigcap_{k=1}^{\infty} E_k$ and, for each $k$, $A \subset \text{interior } E_k$.

Let $M$ be an $m$-dimensional compact metric space and let $F$ be a set-valued function on $M$ such that:

(i) for each $x \in M$, $F(x)$ is a cellular subset of $S^n$, and

(ii) $F$ is upper semi-continuous.

A covering pair for $F$ and $M$ is an ordered pair $(G, D)$ such that:

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(i) $G$ is a finite open covering of $M$,
(ii) $D$ is a function with domain $G$ such that for each $U \in G$, $D(U)$ is a topological $n$-cell which is contained in $S^n$, and
(iii) for each $x \in M$, if $x \in U \in G$ then $F(x) \subseteq \text{interior } D(U)$.

**Lemma 1.** There exists a covering pair.

**Proof.** For each $x \in M$, there exists a topological $n$-cell $\Delta(x)$ such that $F(x) \subseteq \text{interior } \Delta(x)$ and $\Delta(x) \subseteq S^n$. For each $x \in M$, there exists a neighborhood $V(x)$ such that if $t \in V(x)$ then $F(t) \subseteq \text{interior } \Delta(x)$. \{ $V(x) | x \in M$ \} is an open covering of $M$, and, since $M$ is compact, this covering has a finite subcovering $G = \{ V(x_1), \ldots, V(x_k) \}$. We define $D(V(x_j)) = \Delta(x_j)$ for $j = 1, \ldots, k$. It is easy to verify that $(G, D)$ is a covering pair.

**Lemma 2.** If $(G, D)$ is a covering pair, then there exists a covering pair $(G^*, D^*)$ such that:
(i) $G^*$ is a star refinement of $G$, and
(ii) if $U \in G$, $U^* \in G^*$ and $U^* \subseteq U$, then $D^*(U^*) \subseteq D(U)$.

**Proof.** Let $\lambda$ be the Lebesgue number of the covering $G$. For each $x \in M$, there is a topological $n$-cell $\Delta(x)$ such that $F(x) \subseteq \text{interior } \Delta(x)$ and such that if $x \in U \in G$ then $\Delta(x) \subseteq D(U)$. For each $x \in M$, there is a neighborhood $W(x)$ of $x$ such that:
(i) $W(x)$ is contained in the $\lambda/3$-neighborhood of $x$, and
(ii) if $t \in W(x)$ then $F(t) \subseteq \text{interior } \Delta(x)$. The set \{ $W(x) | x \in M$ \} is an open covering of $M$ and has a finite subcover $G^* = \{ W(x_1), \ldots, W(x_k) \}$. We define $D^*(W(x_j)) = \Delta(x_j)$ for $j = 1, \ldots, k$. It is easy to verify that $(G^*, D^*)$ has the desired properties.

Let $F$ be an upper semi-continuous set-valued function on a compact finite-dimensional metric space $M$ such that, for each $x \in M$, $F(x)$ is a cellular subset of $S^n$.

**Theorem 1.** There exists a single-valued continuous function $g: M \to S^n$ such that $g(x) \subseteq S^n - F(x)$ for each $x \in M$.

**Proof.** Let $m$ be the dimension of $M$. It follows from Lemmas 1 and 2 that there are covering pairs $(G_0, D_0), \ldots, (G_{2m}, D_{2m})$ such that for each $j$, $1 \leq j \leq 2m$: (i) $G_j$ is a star refinement of $G_{j-1}$, and
(ii) if $U_j \subseteq G_j$, $U_{j-1} \subseteq G_{j-1}$ and $U_j \subseteq U_{j-1}$, then $D_j(U_j) \subseteq D_{j-1}(U_{j-1})$.

We choose a finite open covering $G$ of $M$ such that $G$ is of order $m$, $G$ is a star refinement of $G_{2m}$, and no proper subset of $G$ covers $M$. For each integer $j$, $0 \leq j \leq m$, we define $K_j = \{ x | x \in M$ and $x$ is a member of at most $j+1$ members of $G$ $\}$. Each $K_j$ is a closed subset of $M$, and $K_m = M$. 

For each \( V \in G \) and each integer \( j, 0 \leq j \leq 2m \), we select sets \( \phi_j(V) \in G_j \) such that \( \text{St}(V) \subset \phi_{2m}(V) \) and \( \text{St}(\phi_j(V)) \subset \phi_{j-1}(V) \), for \( 1 \leq j \leq 2m \). We define \( D_j(\phi_j(V)) \) for each \( V \in G \) and \( j = 0, \ldots, 2m \).

We are going to define (inductively) for each \( j, 0 \leq j \leq m \), a mapping \( g_j : K_j \to S^n \) such that, for each \( V \in G \),

\[
g_j[V \cap K_j] \subset \text{Cl}[S^n - \Phi_{2j}(V)].
\]

For each \( V \in G \), we choose a point \( p_V \in S^n - \Phi_0(V) \) and define \( g_0(x) = p_V \) for each \( x \in V \cap K_0 \). Since \( V \cap K_0 \) is closed in \( K_0 \) for each \( V \in G \), \( g_0 \) is continuous.

Now suppose \( 0 < j \leq m \) and \( g_{j-1} \) has been defined. Let \( \sigma = \{ V_0, \ldots, V_j \} \) be a set of \( j+1 \) distinct members of \( G \) such that \( V_0 \cap \cdots \cap V_j \neq \emptyset \). We define \( H_\sigma = (V_0 \cap \cdots \cap V_j) \cap K_j \) and \( W_\sigma = K_j - \bigcup \{ V | V \in G - \sigma \} \). Then \( W_\sigma \) is closed in \( K_j \) and \( H_\sigma \) is open relative to \( W_\sigma \). The mapping \( g_{j-1} \) is defined on \( W_\sigma - H_\sigma \) and \( W_\sigma - H_\sigma \) is closed relative to \( W_\sigma \). It is easy to see that

\[
g_{j-1}[W_\sigma - H_\sigma] \subset \bigcup_{r=0}^{j} \text{Cl}[S^n - \Phi_{2r-2}(V_r)].
\]

Since \( \phi_{2j-1}(V) \subset \text{St}(\phi_{j-1}(V_r)) \subset \phi_{2j-2}(V_r) \) for \( r = 0, \ldots, j, \Phi_{2j-1}(V_0) \subset \bigcap_{r=0}^{j} \Phi_{2j-2}(V_r) \). Therefore, \( g_{j-1}[W_\sigma - H_\sigma] \subset \text{Cl}[S^n - \Phi_{2j-1}(V_0)] \).

The set \( \text{Cl}[S^n - \Phi_{2j-1}(V_0)] \) is the union of a topological \((n-1)\)-sphere \( \Sigma \) and one of the components of \( S^n - \Sigma \). It is known (see [1]) that such sets are absolute retracts. Since \( \text{Cl}[S^n - \Phi_{2j-1}(V_0)] \) is an absolute retract, we can extend \( g_{j-1} | (W_\sigma - H_\sigma) \) to a mapping \( \psi_\sigma : W_\sigma \to \text{Cl}[S^n - \Phi_{2j-1}(V_0)] \).

Since \( V_r \subset \text{St}(V_0) \) for \( r = 0, \ldots, j \),

\[
\Phi_{2j}(V_r) \subset \text{St}(\phi_{2j}(V_0)) \subset \Phi_{2j-1}(V_0).
\]

Thus \( \Phi_{2j}(V_r) \subset \Phi_{2j-1}(V_0) \) and the range of \( \psi_\sigma \) is contained in \( \text{Cl}[S^n - \Phi_{2j}(V_r)] \) for \( r = 0, \ldots, j \). Thus \( \psi_\sigma [V_r \cap W_r] \subset \text{Cl}[S^n - \Phi_{2j}(V_r)] \) for \( r = 0, \ldots, j \).

If \( \sigma' \) is a different system of \( j+1 \) members of \( G \) and \( W_{\sigma'} \cap W_\sigma \neq \emptyset \), then \( \psi_{\sigma'}[W_{\sigma'} \cap W_\sigma] = \psi_{\sigma}[W_{\sigma'} \cap W_\sigma] = g_{j-1}[W_{\sigma'} \cap W_\sigma] \). It follows that we can piece the mappings \( \psi_\sigma \) and \( g_{j-1} \) together to obtain a mapping \( g_j : K_j \to S^n \). It is obvious that \( g_j[V \cap K_j] \subset \text{Cl}[S^n - \Phi_{2j}(V)] \) for each \( V \in G \).

We define \( g = g_m \). Since \( K_m = M \), \( g \) is a mapping of \( M \) into \( S^n \) such
that \( g[V] \subseteq \text{Cl}[S^n - \Phi_{2m}(V)] \) for each \( V \in G \). If \( x \in V \in G \), then \( x \in \Phi_{2m}(V) \) and, hence, \( F(x) \subseteq \text{interior} \Phi_{2m}(V) \). It follows that if \( x \in M \), then \( g(x) \in S^n - F(x) \).

3. Homotopy for a class of set-valued functions. Let \( M \) be a finite-dimensional compact metric space. We define \( \Gamma(M, S^n) \) to be the set of all upper semi-continuous set-valued functions \( F \) on \( M \) such that for each \( x \in M \), \( F(x) \) is a cellular subset of \( S^n \). We let \( I = [0, 1] \).

Let \( F \) and \( G \) be members of \( \Gamma(M, S^n) \). A function \( H \) is a cellular homotopy relating \( F \) to \( G \) if:

(i) \( H \in \Gamma(M \times I, S^n) \), and

(ii) for all \( x \in M \), \( H(x, 0) = F(x) \) and \( H(x, 1) = G(x) \).

If there exists a cellular homotopy relating \( F \) to \( G \), then we say that \( F \) is homotopic to \( G \) and write \( F \sim G \). The relation \( \sim \) is an equivalence relation and partitions \( \Gamma(M, S^n) \) into equivalence classes which we call cellular homotopy classes.

Let \( F \in \Gamma(M, S^n) \) and let \( f : M \to S^n \) be a (single-valued) continuous function. A function \( H \) is a special homotopy relating \( F \) to \( f \) if:

(i) \( H \) is a cellular homotopy relating \( F \) to \( f \), and

(ii) for all \( x \in M \) and \( 0 \leq t \leq 1 \), \( H(x, t) \) is homeomorphic to \( F(x) \).

If \( F \) is a single-valued function as well as \( f \), then (ii) implies that \( H \) is single-valued, and since upper semi-continuity is equivalent to continuity for single-valued functions, in this case \( H \) is an ordinary homotopy.

**Lemma 3.** If \( F \in \Gamma(M, S^n) \), then there exists a single-valued continuous function \( f : M \to S^n \) and a special homotopy \( H \) relating \( F \) to \( f \).

**Proof.** For each \( p \in S^n \), we define a mapping \( J_p : [S^n - p] \times I \to S^n \) by

\[
J_p(x, t) = \left(-tp + (1 - t)x\right)/\| -tp + (1 - t)x\|
\]

for \( x \in S^n - p \), \( 0 \leq t \leq 1 \). \( J_p \) is a pseudo-isotopy, since the map \( \phi_t \) defined by \( \phi_t(x) = J_p(x, t) \) is a homeomorphism on \( S^n - p \) if \( 0 \leq t < 1 \), \( \phi_0 \) is the identity mapping on \( S^n - p \), and \( \phi_1 \) is the constant map which takes \( S^n - p \) into \( -p \).

By Theorem 1, there is a mapping \( g : M \to S^n \) such that \( g(x) \subseteq S^n - F(x) \) for each \( x \in M \). We define \( f(x) = -g(x) \) and

\[
H(x, t) = \{J_{g(x)}(y, t) \mid y \in F(x)\}
\]

for \( x \in M \) and \( 0 \leq t \leq 1 \). Obviously, \( f \) is a continuous function on \( M \) into \( S^n \), and it is easy to verify that \( H \) is a special homotopy relating \( F \) to \( f \).
Theorem 2. Each cellular homotopy class of $\Gamma(M, S^n)$ contains a single-valued continuous function $f: M \to S^n$.

Proof. This result follows immediately from Lemma 3 and the fact that special homotopies are cellular homotopies.

Our final theorem shows that the notion of cellular homotopy which we have defined for $\Gamma(M, S^n)$ is a true extension of the usual notion of homotopy for single-valued functions.

Theorem 3. If $f_0$ and $f_1$ are single-valued continuous functions on $M$ into $S^n$ and $H \in \Gamma(M \times I, S^n)$ is a cellular homotopy relating $f_0$ to $f_1$, then there exists a single-valued homotopy $h: M \times I \to S^n$ which relates $f_0$ to $f_1$ in the usual sense.

Proof. We apply Lemma 3 (replacing $M$ by $M \times I$ and $F$ by $H$) to obtain a single-valued continuous function $\phi: M \times I \to S^n$ and a special homotopy $K \in \Gamma((M \times I) \times I, S^n)$ relating $H$ to $\phi$. Now, for $(x, t) \in M \times I$, we define

$$h(x, t) = \begin{cases} K(x, 0, 3t) & \text{if } 0 \leq t \leq 1/3, \\ K(x, 3t - 1, 1) & \text{if } 1/3 \leq t \leq 2/3, \\ K(x, 1, 3 - 3t) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

If $0 \leq t < 1/3$, then $h(x, t) = K(x, 0, 3t)$ is homeomorphic to $K(x, 0, 0) = H(x, 0) = f_0(x)$ and, hence, is a one-point set. Likewise, if $2/3 < t \leq 1$, then $h(x, t)$ is a one-point set. If $1/3 \leq t \leq 2/3$, then $h(x, t) = K(x, 3t - 1, 1) = (x, 3t - 1)$ and since $\phi$ is single-valued, $h(x, t)$ is a one-point set. Thus $h$ is a single-valued function on $M \times I$ into $S^n$.

Since $K$ is upper semi-continuous, $h$ is also upper semi-continuous. Since $h$ is single-valued, this implies that $h$ is continuous.

We have shown that $h: M \times I \to S^n$ is an ordinary single-valued homotopy. Since $h(x, 0) = K(x, 0, 0) = H(x, 0) = f_0(x)$ and $h(x, 1) = K(x, 1, 0) = H(x, 1) = f_1(x)$, $h$ relates $f_0$ to $f_1$ in the usual sense.

It should be remarked that it can be shown that the smallest equivalence relation containing both homotopies of single-valued functions and special homotopies is the relation generated by cellular homotopies. The proof is similar to that of Theorem 4.

Bibliography


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