ON RESTRICTIONS OF FUNCTIONS IN THE SPACES $P_{a,p}$ AND $B_{a,p}$

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In this note we give a generalization of a result of Aronszajn and Smith [1] concerning sections of exceptional sets for the spaces of Bessel potentials on $R^n$ of $L^2$ functions and restrictions of functions in these spaces. For the sake of completeness we recall briefly the relevant definitions and theorems. We refer for details to [2]. Throughout this paper we will be concerned with functions defined on the space $R^n$; we write $f$ for $f_{R^n}$, $L^p$ for $L^p(R^n)$, $C_0^\infty$ for $C_0^\infty(R^n)$, etc.

The Bessel kernel $G_a$ on $R^n$ of order $\alpha>0$ is defined as the inverse Fourier transform of the function

$$G_a(x) = \frac{(2\pi)^{-n/2}}{(1 + |x|^2)^{\alpha/2}}.$$  

$G_a(x)$ is positive for all $x \in R^n$, $x \neq 0$, also $\|G_a\|_1 = \int G_a(x) \, dx = 1$. For $\alpha > 0$ and $f \in L^1_\text{loc}$ we denote $(G_a f)(x) = (G_a * f)(x) = \int G_a(x-y)f(y) \, dy$, if the integral exists; we also define the operator $G_0$ as the identity operator.

For $1 \leq p < \infty$ and $\alpha > 0$, $\mathcal{A}_{a,p}$ denotes the class of all sets $A \subset R^n$ for which there exists a function $f \in L^p$, $f \geq 0$, such that $A \subset \{x: (G_a f)(x) = +\infty\}$. For $\alpha = 0$ we define $\mathcal{A}_{0,p} = \mathcal{A}_0$ as the class of sets of $n$-dimensional Lebesgue measure 0. $\mathcal{A}_{a,p}$ is obviously hereditary; it can be proved to be $\sigma$-additive. If $f \in L^p$, then the integral $(G_a f)(x)$ exists and is finite exc. $\mathcal{A}_{a,p}$ (i.e., outside of a set $A \in \mathcal{A}_{a,p}$); we denote by $P_{a,p} = P_{a,p}(R^n)$, $\alpha > 0$, the class of all functions $u(x)$ defined exc. $\mathcal{A}_{a,p}$ by the formula $u(x) = (G_a f)(x)$ with $f$ running over $L^p(R^n)$. For $\alpha = 0$ we put $P_{0,p} = L^p$. For $u \in P_{a,p}$, $u = G_0 f$, we define the norm $\|u\|_{a,p} = \|G_a f\|_{a,p} = \|f\|_p$. With this norm and the exceptional class $\mathcal{A}_{a,p}$, $P_{a,p}$ becomes a complete functional space (i.e., every sequence in $P_{a,p}$ convergent in norm contains a subsequence convergent exc. $\mathcal{A}_{a,p}$).

It can be proved that $P_{a,p}$ is the perfect functional completion (i.e., the functional completion relative to the smallest exceptional class) of $C_0^\infty$ with the norm $\|\cdot\|_{a,p}$; on $C_0^\infty$ the norm $\|\cdot\|_{a,p}$ can be given by the more explicit expression $\|u\|_{a,p} = \|G_{2m-a}(1-\Delta)^m u\|_p$, where $m$ is any integer such that $2m > \alpha$ and $\Delta$ is the Laplace operator.

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The following proposition gives an equivalent characterization of the class $\mathcal{A}_{a,p}$.

**Proposition 1.** $A \in \mathcal{A}_{a,p}$ if and only if there exists a sequence $\{u_i\} \subset C_0^\infty$, $u_i \geq 0$, Cauchy in the norm $\|\cdot\|_{a,p}$ and such that $\lim_{i \to \infty} u_i(x) = +\infty$ for all $x \in A$.

For $\alpha > 0$, $k$ an integer, $k > \alpha$ and for $u \in L^p$, $1 \leq p < \infty$, we define

$$
(\|u\|_{a,p,k})^p = \|u\|_{L^p}^p + \int_{R^n} |\Delta_k u(x)|^p \cdot |t|^{-n-p\alpha} \, dt,
$$

where $\Delta_k$ denotes the $k$th forward difference with increment $t \in R^n$. It can be proved that for two integers $k$, $k_1$, $k > \alpha$, $k_1 > \alpha$, the norms $\|\cdot\|_{a,p,k}$, $\|\cdot\|_{a,p,k_1}$ are equivalent on the subspace of $L^p$ where they are finite. Denote this subspace by $\mathcal{B}^{a,p}$. Let $0 < \epsilon < \min(1, \alpha)$ and denote by $\mathcal{B}_{a,p}$ the class of all sets $A \subset R^n$ for which there exists a function $f \geq 0$, $f \in \mathcal{B}^{a,p}$ such that $A \subset \{ x : (G_{a-f})(x) = +\infty \}$. $\mathcal{B}_{a,p}$ is clearly hereditary; it can be shown to be $\sigma$-additive and independent of the choice of $\epsilon$. If $f \in \mathcal{B}^{a,p}$, then the function $u(x) = (G_{a-f})(x)$ is defined and finite exc. $\mathcal{B}_{a,p}$. Denote by $B^{a,p}$ the class of all functions $u$ such that $u(x) = (G_{a-f})(x)$ exc. $\mathcal{B}_{a,p}$ for some $f \in \mathcal{B}^{a,p}$. It can be proved that $B^{a,p}$ with one of the equivalent norms $\|\cdot\|_{a,p,k}$, $k > \alpha, a > 0$, is a complete functional space rel. $\mathcal{B}_{a,p}$; moreover, it coincides with the perfect functional completion of $C_0^\infty(R^n)$ with the norm $\|\cdot\|_{a,p,k}$. The perfect functional completion being unique, if it exists, it follows that $B^{a,p}$ does not depend on $\epsilon$ occurring in the definition.

For $p = 2$, both $P^{a,p}$ and $B^{a,p}$ coincide with the space $P^a$ of Bessel potentials of $L^2$ functions.

The following proposition will be useful later.

**Proposition 2.** The convolution with the kernel $G_{\beta}$, $\beta > 0$, establishes a bounded isomorphism of the space $B^{a,p}$ onto $B^{a+\beta,p}$. More precisely, if $u \in B^{a,p}$, then $G_{\beta}u = v \in B^{a+\beta,p}$ and every $v \in B^{a+\beta,p}$ is of this form; for fixed $\beta > 0$, $k > \alpha$, $k_1 > \alpha + \beta$, there are two constants $c_1 > 0$, $c_2 > 0$ such that

$$
c_1\|u\|_{a,p,k} \leq \|v\|_{a+\beta,p,k_1} = \|G_{\beta}u\|_{a+\beta,p,k_1} \leq c_2\|u\|_{a,p,k},
$$

for all $u \in B^{a,p}$.

We also state the counterpart of Proposition 1 for the class $\mathcal{B}_{a,p}$.

**Proposition 3.** $A \in \mathcal{B}_{a,p}$, $\alpha > 0$, $1 \leq p < \infty$, if and only if there exists a sequence $\{u_i\} \subset C_0^\infty$, $u_i \geq 0$, Cauchy in the norm $\|\cdot\|_{a,p,k}$, $k > \alpha$, and such that $\lim_{i \to \infty} u_i(x) = +\infty$ for all $x \in A$.
Let $n', n''$ be two positive integers, $n = n' + n''$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we write $x = (x_1, \cdots, x_{n'}, x_{n'+1}, \cdots, x_n) = (x', x'')$, and for a fixed $x' \in \mathbb{R}^{n'}$ denote by $A_{x'}$ the section of a set $A \subset \mathbb{R}^n$, $A_{x'} = \{ x'' \in \mathbb{R}^{n''}: (x', x'') \in A \}$ and by $u_{x'}$ the restriction of a function $u$ defined on $\mathbb{R}^n$, $u_{x'}(x'') = u(x', x'')$. Denote further by $\mathcal{A}_{a,p}$, $\mathcal{B}_{a,p}$, $\mathcal{A}_{a,p}$, $\mathcal{B}_{a,p}$ the exceptional classes for $P^{a,p}(\mathbb{R}^{n'})$, $B^{a,p}(\mathbb{R}^{n'})$, $P^{a,p}(\mathbb{R}^{n''})$, and $B^{a,p}(\mathbb{R}^{n''})$ and by $G_{a'}$, $G_a$ the Bessel kernels of order $a$ on the spaces $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$, respectively.

The following theorem gives a description of sections $A_{x'}$ of sets $A$ in $\mathcal{A}_{a,p}$ and $\mathcal{B}_{a,p}$ and of restrictions $u_{x'}$ of functions $u$ in $P^{a,p}$ and $B^{a,p}$.

**Theorem.** Let $1 < p < \infty$ and $\alpha > 0$.

(i) If $A \in \mathcal{A}_{a,p}$, then $A_{x'} \in \mathcal{B}_{a-p}$ for all $\beta$, $0 \leq \beta \leq \alpha$; if $u \in P^{a,p}(\mathbb{R}^{n'})$, then $u_{x'} \in P^{a-p}(\mathbb{R}^{n''})$ for all $\beta$, $0 \leq \beta \leq \alpha$.

(ii) If $A \in \mathcal{B}_{a,p}$, then $A_{x'} \in \mathcal{B}_{a-p}$ for all $\beta$, $0 < \beta < \alpha$; also $A_{x'} \in \mathcal{B}_{a-p}$ for all $\beta$, $0 \leq \beta < \alpha$. If $u \in B^{a,p}(\mathbb{R}^{n'})$, then $u_{x'} \in B^{a-p}(\mathbb{R}^{n''})$ for all $0 < \beta \leq \alpha$ and $u_{x'} \in P^{a-p}(\mathbb{R}^{n''})$ for $0 \leq \beta < \alpha$.

Before giving the proof we make the following remark.

**Remark.** In the case when $\alpha > n'/p$ it is known that $u \in B^{a,p}(\mathbb{R}^{n'})$ implies $u_{x'} \in B^{a-n'/p}(\mathbb{R}^{n''})$ for all $x' \in \mathbb{R}^{n'}$. The only information we get from (ii) is that $u_{x'} \in B^{a-n'/p}(\mathbb{R}^{n''})$ for $\mathcal{A}_{a/p,p}$, which shows that in this case the result of (ii) is not the best possible: the class $\mathcal{A}_{a/p,p}$ is not empty (for $p = 2$ it is the class of the sets of logarithmic capacity 0), although $\mathcal{B}_{a,p}$ is empty for all $\beta > n'/p$.

The proof of the theorem depends on the following lemmas

**Lemma 1.** For $f \in L^p(\mathbb{R}^{n'})$ and $0 < \epsilon < 1$ let

\[ f'(x') = \left( \int_{\mathbb{R}^{n''}} \left| f(x', x'') \right|^p dx'' \right)^{1/p}, \]

\[ f'_\epsilon(x') = \left( \int_{\mathbb{R}^{n''}} \int_{\mathbb{R}^{n''}} \left| f(x', x'', t'', t') \right|^p \frac{dx'dt'}{|t''|^{n''+p\epsilon}} \right)^{1/p}. \]

Then (i) $f' \in L^p(\mathbb{R}^{n'})$ and $\|f'\|_{L^p} = \|f\|_{L^p(\mathbb{R}^{n'})}$,

(ii) $f \in B^{a,p}(\mathbb{R}^{n'})$ implies $f' \in B^{a-p}(\mathbb{R}^{n'})$, $f'_\epsilon \in L^p(\mathbb{R}^{n'})$ and $\|f'_\epsilon\|_{L^p} \leq C\|f\|_{L^p}$, with a constant $C$ depending only on $\epsilon$, $n'$, $n''$.

By $\|f\|_{L^p}$, $\|f'\|_{L^p}$, $\|f'_\epsilon\|_{L^p}$ we understand the norms of $f'$ as a function on $\mathbb{R}^{n'}$.

**Proof.** (i) is trivial. To prove the first part of (ii) we only need
an estimate of the second term in the definition of the norm (2). We have
\[ |t'|^{-\sigma} |\Delta_t f'(x')| \]
\[ = |t'|^{-\sigma} \left| \|f(x' + t', x'')\|_{L^p(\mathbb{R}^{n''})} - \|f(x', x'')\|_{L^p(\mathbb{R}^{n''})} \right| \]
\[ \leq \|f(x' + t', x'') - f(x', x'')\|_{L^p(\mathbb{R}^{n''})}. \]
(3)

Also, for \(|t'| \neq 0\),
\[ \int_{\mathbb{R}^{n''}} |t|^{-n - \epsilon_p} dt'' = \int_{\mathbb{R}^{n''}} \left( |t'|^2 + |t''|^2 \right)^{(n + \epsilon_p)/2} dt'' \]
\[ = |t'|^{-n - \epsilon_p} \int_{\mathbb{R}^{n'}} \left( 1 + |y''|^2 \right)^{(n + \epsilon_p)/2} dy'' \]
\[ = C_1 |t'|^{-n - \epsilon_p}. \]
(4)

From (3) and (4) we get
\[ \left( \int_{\mathbb{R}^{n}} \|\Delta_t f'\|_{L^p}^p |t|^{-n - \epsilon_p} dt't' \right)^{1/p} \]
\[ \leq \left( C_1^{-1} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \left| f(x' + t', x'') - f(x', x'') \right|^p \frac{dx'dt}{|t|^{n + \epsilon_p}} \right)^{1/p} \]
\[ \leq C_1^{-1/p} \left[ \left( \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \left| f(x' + t', x'') - f(x' + t'/2, x'' + t''/2) \right|^p \frac{dx'dt}{|t|^{n + \epsilon_p}} \right)^{1/p} \right. \]
\[ + \left. \left( \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \left| f(x' + t'/2, x'' + t''/2) - f(x', x'') \right|^p \frac{dx'dt}{|t|^{n + \epsilon_p}} \right)^{1/p} \right] \]
\[ = 2^{1-p} C_{1/p} \left( \int_{\mathbb{R}^{n}} \|\Delta_t f'\|_{L^{p-\epsilon_p,n}}^p dt \right)^{1/p}. \]

Hence, \(\|f'\|_{\epsilon, p, 1} \leq \max(1, 2^{1-p} C_{1/p}) \|f\|_{\epsilon, p, 1}.\)

The proof of the second part of (ii) is obtained by interchanging the roles of \(x'\) and \(x''.\)

As usual \(S\) denotes the class of all \(C^\infty\) functions of rapid decrease.

**Lemma 2.** Let \(\gamma \geq 0, \delta \geq 0, 1 < p < \infty\) be fixed. The equation
\[ \int_{\mathbb{R}^{n}} G_{\gamma + \delta}(x - y) g(y) dy \]
\[ = \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n'}} G_\gamma'(x' - y') G_\delta''(x'' - y'') f(y', y'') dy'dy'' \]
\[ (5) \]
establishes a 1-1 mapping \( f \to g \) of \( S \) onto itself. There exists a constant \( C \) such that

1. \( \|f\|_{\eta, p, k} \leq C \|g\|_{\eta, p, k} \), for all \( g \in S \), \( 0 < \eta < k \).

**Proof.** Rewriting (5) in terms of Fourier transforms and using (1) we get

\[
\hat{f}(\xi) = \frac{(1 + |\xi|^2)^{\gamma/2}(1 + |\xi|^2)^{\beta/2}}{(1 + |\xi|^2)^{\gamma+\beta/2}} \hat{g}(\xi).
\]

The first statement of the lemma is now trivial. Using the Mihlin theorem [3] about multipliers of Fourier transforms we verify immediately that the coefficient of \( \hat{g}(\xi) \) on the right hand side of (6) is a multiplier of type \((p, p)\) for every \( p, 1 < p < \infty \), which proves (i).

(ii) is obtained from (i) by replacing \( f \) and \( g \) by \( \Delta f \) and \( \Delta g \), respectively, with an arbitrary fixed \( t \in \mathbb{R}^n \).

**Proof of the Theorem.** Let \( \gamma \geq 0 \), \( \delta \geq 0 \) and \( u, f, g \in S \) be related by the equation

\[
u(x', x'') = \int_{\mathbb{R}^n} G_{x+y}(x-y)g(y) \, dy
\]

Let \( \alpha > 0 \) and \( 0 \leq \beta \leq \alpha \); we put, in (7), \( \delta = \beta, \gamma = \alpha - \beta \). By the definition of the norm \( \| \cdot \|_{\beta, p} \), we have, for every \( x' \in \mathbb{R}^n \),

\[
\|u(x')\|_{\beta, p} = \left\| \int_{\mathbb{R}^n} G_{x-y}(x'-y)f(y', y'') \, dy' \right\|_{L^p}
\]

and using the continuous version of the Minkowski inequality

\[
\|u(x')\|_{\beta, p} \leq \int_{\mathbb{R}^n} G_{x-y}(x'-y)\|f(y', y'')\|_{L^p(\mathbb{R}^n', \mathbb{R}^n''')} \, dy'
\]

Let now \( \alpha > 0 \), \( 0 < \beta \leq \alpha \) and \( \epsilon = (1/2) \) min\((1, \beta)\). Choose, in (7), \( \delta = \beta - \epsilon, \gamma = \alpha - \beta \). Using Proposition 2, we get, with an integer \( k > \beta \) and a constant \( C \) depending only on \( \beta \), and \( n'' k \),

\[
\|u(x')\|_{\beta, p, k} \leq C \left\| \int_{\mathbb{R}^n} G_{x-y}(x'-y)f(y', y'') \, dy' \right\|_{\epsilon, p, 1},
\]
and, using again a continuous form of the Minkowski inequality,
\[ ||u_x||_{\beta,p,k} \leq C \left[ \left( \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y') f'(y') \, dy' \right)^p + \left( \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y') f''(y') \, dy' \right)^p \right]^{1/p} \]
(9)

\[ \leq C \left[ \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y') \left[ f'(y') + f''(y') \right] \, dy' \right]. \]

Finally, let \( \alpha > 0, 0 \leq \beta < \alpha \) and \( \epsilon = (1/2) \min(1, \alpha - \beta) \). Put, in (7), \( \delta = \beta, \gamma = \alpha - \beta - \epsilon \). We get, similarly, as in (8),
\[ ||u_x||_{\beta,p} \leq \int_{\mathbb{R}^n} G_{\alpha-\beta,\epsilon}(x' - y') f'(y') \, dy'. \]
(10)

In the inequalities (8), (9), and (10), \( f' \) and \( f'' \) are defined as in Lemma 1.

We shall now prove (ii). Let \( A \in \mathfrak{B}_{\alpha,p} \). By Proposition 3, there is a sequence \( \{u_i\} \subset C_0^\infty, u_i \geq 0, \) Cauchy in \( B^{\alpha,p} \) and such that \( \lim_{i \to \infty} u_i(x) = +\infty \) for all \( x \in A \). Let \( \epsilon = (1/2) \min(1, \beta) \) and \( \{f_i\}, \{g_i\} \subset S \) be such that
\[ u_i(x) = \int_{\mathbb{R}^n} G_{\alpha-\epsilon}(x - y) g_i(y) \, dy \]
(11)
\[ = \int_{\mathbb{R}^n} G_{\alpha-\epsilon}(x' - y') \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y') f_i(y', y'') \, dy' \, dy''. \]

By Proposition 2, we may assume, without loss of generality, that \( \sum_{i=1}^{\infty} ||g_{i+1} - g_i||_{\epsilon,p,k} < \infty \), consequently, by Lemmas 1 and 2, \( h' = \sum_{i=1}^{\infty} (f_{i+1} - f_i)' + \sum_{i=1}^{\infty} (f_{i+1} - f_i) \) \( \in L^p(\mathbb{R}^n) \). We conclude, using (9), that the sequence \( \{u_i\} \) is Cauchy in \( B^{\beta,p}(R^n) \) for every \( x' \) outside of the set \( A' = \{ x' \in R^n : \int_{R^n} G_{\alpha-\beta}(x' - y') h'(y') \, dy' = +\infty \} \in \mathfrak{W}_{\alpha-\beta,p} \).

This proves, using the functional space property, that \( A_x \in \mathfrak{B}_{\alpha,p} \) for \( x' \in A' \) i.e., \( \mathfrak{W}_{\alpha-\beta,p} \).

Let now \( u \in B^{\beta,p} \) and choose an integer \( k > \alpha \). By the functional space property there is a sequence \( \{u_i\} \subset C_0^\infty \) such that \( \lim_{i \to \infty} ||u - u_i||_{\alpha,p,k} = 0 \) and \( \lim_{i \to \infty} u_i(x) = u(x) \) exc. \( \mathfrak{B}_{\alpha,p} \). Let \( A \in \mathfrak{B}_{\alpha,p} \) be the union of the exceptional set of \( u \) (i.e., the set where \( u \) is not defined or infinite) and the set where \( \{u_i(x)\} \) does not converge to \( u(x) \).

Let \( A' \subset \mathfrak{W}_{\alpha-\beta,p} \) be the set with the property that \( A_x \in \mathfrak{B}_{\alpha,p} \) for \( x' \in A' \). Assume without loss of generality that \( \sum_{i=1}^{\infty} ||u_{i+1} - u_i||_{\alpha,p,k} < \infty \), define \( \{f_i\}, \{g_i\} \subset S \) as in (11) and let
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\[ h' = \sum_{i=1}^{\infty} (f_{i+1} - f_i)' + \sum_{i=1}^{\infty} (f_{i+1} - f_i)' \in L^p(\mathbb{R}^n). \]

It follows from (9) that \((u_i)_{x'}\) is Cauchy in \(B^{\beta,p}(\mathbb{R}^n)\) for every \(x'\) outside of the set \(B' = \{ x' \in \mathbb{R}^n : \int G_{a_\beta}(x' - y')h'(y')\,dy' = + \infty \} \). Consequently, for \(x' \notin A' \cup B'\), \((u_1)_{x'}\) is Cauchy in \(B^{\beta,p}(\mathbb{R}^n)\) and converges to \(u_{x'}\) exc. \(\mathcal{B}''\). Since \(A' \cup B' \in \mathcal{H}_{a_\beta,p}\) this proves that \(u_{x'} \in B^{\beta,p}(\mathbb{R}^n)\) exc. \(\mathcal{H}_{a_\beta,p}\). The proofs of the remaining part of (ii) and of (i) follow the same idea and are even simpler. We use the statements (i) of Lemmas 1 and 2 and inequality (8) to prove (i) and the statements (ii) of Lemmas 1, 2 and inequality (10) to prove the remainder of (ii).

REFERENCES


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