In this note we give a generalization of a result of Aronszajn and Smith [1] concerning sections of exceptional sets for the spaces of Bessel potentials on \( R^n \) of \( L^2 \) functions and restrictions of functions in these spaces. For the sake of completeness we recall briefly the relevant definitions and theorems. We refer for details to [2]. Throughout this paper we will be concerned with functions defined on the space \( R^n \); we write \( f \) for \( f\|_{L^p(R^n)} \), \( C_0^\infty \) for \( C_0^\infty(R^n) \), etc.

The Bessel kernel \( G_\alpha \) on \( R^n \) of order \( \alpha > 0 \) is defined as the inverse Fourier transform of the function

\[
\hat{G}_\alpha(\xi) = \frac{(2\pi)^{-n/2}}{\left(1 + |\xi|^2\right)^{n/2}}.
\]

\( G_\alpha \) is positive for all \( x \in R^n, x \neq 0 \), also \( \|G_\alpha\|_L^1 = \int G_\alpha(x) \, dx = 1 \). For \( \alpha > 0 \) and \( f \in L^1_\infty \) we denote \( (G_\alpha f)(x) = (G_\alpha \ast f)(x) = \int G_\alpha(x-y)f(y) \, dy \), if the integral exists; we also define the operator \( G_\alpha \) as the identity operator.

For \( 1 \leq p < \infty \) and \( \alpha > 0 \), \( \mathcal{A}_{\alpha,p} \) denotes the class of all sets \( A \subset R^n \) for which there exists a function \( f \in L^p, f \geq 0 \), such that \( A \subset \{ x : (G_\alpha f)(x) = +\infty \} \). For \( \alpha = 0 \) we define \( \mathcal{A}_{0,p} = \mathcal{A}_0 \) as the class of sets of \( n \)-dimensional Lebesgue measure 0. \( \mathcal{A}_{\alpha,p} \) is obviously hereditary; it can be proved to be \( \sigma \)-additive. If \( f \in L^p \), then the integral \( (G_\alpha f)(x) \) exists and is finite exc. \( \mathcal{A}_{\alpha,p} \) (i.e., outside of a set \( A \subset \mathcal{A}_{\alpha,p} \)); we denote by \( P_{\alpha,p} = P_{\alpha,p}(R^n), \alpha > 0 \), the class of all functions \( u(x) \) defined exc. \( \mathcal{A}_{\alpha,p} \) by the formula \( u(x) = (G_\alpha f)(x) \) with \( f \) running over \( L^p(R^n) \). For \( \alpha = 0 \) we put \( P_{0,p} = L^p \). For \( u \in P_{\alpha,p}, u = G_\alpha f \), we define the norm \( \|u\|_{\alpha,p} = \|G_\alpha f\|_{\alpha,p} = \|f\|_{L^p} \). With this norm and the exceptional class \( \mathcal{A}_{\alpha,p} \), \( P_{\alpha,p} \) becomes a complete functional space (i.e., every sequence in \( P_{\alpha,p} \) convergent in norm contains a subsequence convergent exc. \( \mathcal{A}_{\alpha,p} \)). It can be proved that \( P_{\alpha,p} \) is the perfect functional completion (i.e., the functional completion relative to the smallest exceptional class) of \( C_0^\infty \) with the norm \( \| \cdot \|_{\alpha,p} \); on \( C_0^\infty \) the norm \( \| \cdot \|_{\alpha,p} \) can be given by the more explicit expression \( \|u\|_{\alpha,p} = \|G_{2m-\alpha}(1-\Delta)^{m}u\|_{L^p} \), where \( m \) is any integer such that \( 2m > \alpha \) and \( \Delta \) is the Laplace operator.

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The following proposition gives an equivalent characterization of the class $\mathcal{A}_{a,p}$.

**Proposition 1.** A $\in \mathcal{A}_{a,p}$ if and only if there exists a sequence $\{u_n\} \subset C_0^\infty$, $u_n \geq 0$, Cauchy in the norm $\| \cdot \|_{a,p}$ and such that $\lim_{n \to \infty} u_n(x) = +\infty$ for all $x \in A$.

For $a > 0$, $k$ an integer, $k > a$ and for $u \in L^p$, $1 \leq p < \infty$, we define

$$\| u \|_{a,p,k} = \| u \|_p + \int_{\mathbb{R}^n} \| \Delta_k u \|_p \| t \|^{-n-a} dt,$$

where $\Delta_k$ denotes the $k$th forward difference with increment $t \in \mathbb{R}^n$. It can be proved that for two integers $k$, $k_1$, $k > a$, $k_1 > a$, the norms $\| \cdot \|_{a,p,k}$, $\| \cdot \|_{a,p,k_1}$ are equivalent on the subspace of $L^p$ where they are finite. Denote this subspace by $\mathcal{A}_{a,p}$. Let $0 < \epsilon < \min(1, a)$ and denote by $\mathcal{B}_{a,p}$ the class of all sets $A \subset \mathbb{R}^n$ for which there exists a function $f \geq 0$, $f \in \mathcal{B}_{a,p}$ such that $A = \{ x : (G_{a-\epsilon} f)(x) = +\infty \}$. $\mathcal{B}_{a,p}$ is clearly hereditary; it can be shown to be $\sigma$-additive and independent of the choice of $\epsilon$. If $f \in \mathcal{B}_{a,p}$, then the function $u(x) = (G_{a-\epsilon} f)(x)$ is defined and finite exc. $\mathcal{B}_{a,p}$. Denote by $B_{a,p}$ the class of all functions $u$ such that $u(x) = (G_{a-\epsilon} f)(x)$ exc. $\mathcal{B}_{a,p}$ for some $f \in \mathcal{B}_{a,p}$. It can be proved that $B_{a,p}$ with one of the equivalent norms $\| \cdot \|_{a,p,k}$, $k > a > 0$, is a complete functional space rel. $\mathcal{B}_{a,p}$; moreover, it coincides with the perfect functional completion of $C_0^\infty(\mathbb{R}^n)$ with the norm $\| \cdot \|_{a,p,k}$. The perfect functional completion being unique, if it exists, it follows that $B_{a,p}$ does not depend on $\epsilon$ occurring in the definition.

For $p = 2$, both $P_{a,p}$ and $B_{a,p}$ coincide with the space $P_a$ of Bessel potentials of $L^2$ functions.

The following proposition will be useful later.

**Proposition 2.** The convolution with the kernel $G_{a,\beta}$, $\beta > 0$, establishes a bounded isomorphism of the space $B_{a,p}$ onto $B_{a+\beta,p}$. More precisely, if $u \in B_{a,p}$, then $G_{a,\beta} u = v \in B_{a+\beta,p}$ and every $v \in B_{a+\beta,p}$ is of this form; for fixed $\beta > 0$, $k > a$, $k_1 > a + \beta$, there are two constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \| u \|_{a,p,k} \leq \| v \|_{a+\beta,p,k_1} = \| G_{a,\beta} u \|_{a+\beta,p,k_1} \leq c_2 \| u \|_{a,p,k}$$

for all $u \in B_{a,p}$.

We also state the counterpart of Proposition 1 for the class $\mathcal{B}_{a,p}$.

**Proposition 3.** $A \in \mathcal{B}_{a,p}$, $a > 0$, $1 \leq p < \infty$, if and only if there exists a sequence $\{u_n\} \subset C_0^\infty$, $u_n \geq 0$, Cauchy in the norm $\| \cdot \|_{a,p,k}$, $k > a$, and such that $\lim_{n \to \infty} u_n(x) = +\infty$ for all $x \in A$. 

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Let \( n', n'' \) be two positive integers, \( n = n' + n'' \). For \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \) we write \( x = (x_1, \cdots, x_{n'-1}, \cdots, x_n) = (x', x'') \), and for a fixed \( x' \in \mathbb{R}^{n'} \) denote by \( A_{x'} \) the section of a set \( A \subset \mathbb{R}^n \), \( A_{x'} = \{ x'' \in \mathbb{R}^{n''} : (x', x'') \in A \} \) and by \( u_{x'} \) the restriction of a function \( u \) defined on \( \mathbb{R}^n \), \( u_{x'}(x'') = u(x', x'') \). Denote further by \( \mathcal{A}_{\alpha,p}, \mathcal{B}_{\alpha,p}, \mathcal{A}_{\alpha}^{n'}, \mathcal{B}_{\alpha}^{n'} \) the exceptional classes for \( \mathcal{P}_{\alpha,p}(\mathbb{R}^n) \), \( \mathcal{P}_{\alpha}(\mathbb{R}^n) \), \( \mathcal{P}_{\alpha}(\mathbb{R}^{n''}) \), \( \mathcal{B}_{\alpha}(\mathbb{R}^{n''}) \) and by \( G_{\alpha}, G_{\alpha}' \) the Bessel kernels of order \( \alpha \) on the spaces \( \mathbb{R}^n \) and \( \mathbb{R}^{n''} \), respectively.

The following theorem gives a description of sections \( A_{x'} \) of sets \( A \) in \( \mathcal{A}_{\alpha,p} \) and \( \mathcal{B}_{\alpha,p} \) and of restrictions \( u_{x'} \) of functions \( u \) in \( \mathcal{P}_{\alpha,p} \) and \( \mathcal{B}_{\alpha,p} \).

**Theorem.** Let \( 1 < p < \infty \) and \( \alpha > 0 \).

(i) If \( A \in \mathcal{A}_{\alpha,p} \), then \( A_{x'} \in \mathcal{A}_{\alpha-\beta,p}^{n'} \) for all \( \beta, 0 \leq \beta \leq \alpha \); if \( u \in \mathcal{P}_{\alpha,p}(\mathbb{R}^n) \), then \( u_{x'} \in \mathcal{P}_{\alpha-\beta,p}(\mathbb{R}^{n''}) \) for all \( \beta, 0 \leq \beta \leq \alpha \).

(ii) If \( A \in \mathcal{B}_{\alpha,p} \), then \( A_{x'} \in \mathcal{B}_{\alpha-\beta,p}^{n'} \) for all \( \beta, 0 < \beta \leq \alpha \); also \( A_{x'} \in \mathcal{B}_{\alpha,p}^{n''} \) for all \( \beta, 0 \leq \beta < \alpha \). If \( u \in \mathcal{B}_{\alpha,p}(\mathbb{R}^n) \), then \( u_{x'} \in \mathcal{B}_{\alpha-\beta,p}(\mathbb{R}^{n''}) \) for all \( 0 < \beta \leq \alpha \) and \( u_{x'} \in \mathcal{P}_{\alpha-\beta,p}(\mathbb{R}^{n''}) \) for all \( 0 \leq \beta < \alpha \).

Before giving the proof we make the following remark.

**Remark.** In the case when \( \alpha > n'/p \) it is known that \( u \in \mathcal{B}_{\alpha,p}(\mathbb{R}^n) \) implies \( u_{x'} \in \mathcal{B}_{\alpha-\beta,p}(\mathbb{R}^{n''}) \) for all \( x' \in \mathbb{R}^{n'} \). The only information we get from (ii) is that \( u_{x'} \in \mathcal{B}_{\alpha-\beta,p}(\mathbb{R}^{n''}) \) exc. \( \mathcal{A}_{\alpha-\beta,p}^{n''} \), which shows that in this case the result of (ii) is not the best possible: the class \( \mathcal{A}_{\alpha,p}^{n''} \) is not empty (for \( p = 2 \) it is the class of the sets of logarithmic capacity 0), although \( \mathcal{A}_{\alpha,p}^{n'} \) is empty for all \( \beta > n'/p \).

The proof of the theorem depends on the following lemmas.

**Lemma 1.** For \( f \in L^p(\mathbb{R}^n) \) and \( 0 < \epsilon < 1 \) let

\[
 f'(x') = \left( \int_{\mathbb{R}^{n''}} \left| f(x', x'') \right|^p dx'' \right)^{1/p},
\]

\[
 f''(x') = \left( \int_{\mathbb{R}^{n'}} \int_{\mathbb{R}^{n''}} \frac{\left| f(x', x'' + t'') - f(x', x'') \right|^p}{|t'|^{n''+\epsilon}} dx'' dt'' \right)^{1/p}.
\]

Then (i) \( f' \in L^p(\mathbb{R}^n) \) and \( \|f'\|_{L^p} = \|f\|_{L^p}(\mathbb{R}^n) \),

(ii) \( f \in \mathcal{B}_{\epsilon,p}(\mathbb{R}^n) \) implies \( f' \in \mathcal{B}_{\epsilon-\beta,p}(\mathbb{R}^n) \), \( f'' \in L^p(\mathbb{R}^{n''}) \) and \( \|f''\|_{\epsilon,p,1} \leq \|f\|_{\epsilon,p,1} \), \( \|f''\|_{\epsilon,p,1} \leq \|f\|_{\epsilon,p,1} \), with a constant \( C \) depending only on \( \epsilon, n', n'' \).

By \( \|f\|_{L^p}, \|f'\|_{\epsilon,p,1}, \|f''\|_{\epsilon,p,1} \) we understand the norms of \( f' \) as a function on \( \mathbb{R}^n \).

**Proof.** (i) is trivial. To prove the first part of (ii) we only need...
an estimate of the second term in the definition of the norm (2). We have

\[ |t'|^p \Delta_t f'(x') | \]

\[ = |t'|^p |\int f(x' + t', x'') - f(x', x'')| L^p(\mathbb{R}^n) | \]

\[ \leq \|f(x' + t', x'') - f(x', x'')\| L^p(\mathbb{R}^n), \]

(3)

Also, for \(|t'| \neq 0|\),

\[ \int_{\mathbb{R}^n} |t|^{-n-ep} \, dt'' = \int_{\mathbb{R}^n} (|t'|^2 + |t''|^2)^{-(n+ep)/2} \, dt'' \]

\[ = |t'|^{-n-ep} \int_{\mathbb{R}^n} (1 + |y''|^2)^{-(n+ep)/2} \, dy'' \]

\[ = C_1 |t'|^{-n-ep}. \]

From (3) and (4) we get

\[ \left( \int_{\mathbb{R}^n} \left| \Delta_t f' \right|^p |t|^{-n-ep} \, dt' \right)^{1/p} \]

\[ \leq \left( C_{-1} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| f(x' + t', x'') - f(x', x'') \right|^p \right) \frac{dx'dt}{|t|^{n+ep}} \right)^{1/p} \]

\[ \leq C_{-1}^{-1/p} \left[ \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| f(x' + t', x'') - f(x' + t'/2, x'' + t''/2) \right|^p \right) \frac{dx'dt}{|t|^{n+ep}} \right)^{1/p} \right. \]

\[ + \left. \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| f(x' + t', x'') - f(x', x'') \right|^p \right) \frac{dx'dt}{|t|^{n+ep}} \right)^{1/p} \right] \]

\[ = 2^{1-c} C_{-1}^{-1/p} \left( \int_{\mathbb{R}^n} \left| \Delta_t f' \right|^p e^{-n-ep} \, dt \right)^{1/p}. \]

Hence, \(|f'|_{\ast, p, 1} \leq \max(1, 2^{1-c} C_{-1}^{-1/p})|f|_{\ast, p, 1} . \)

The proof of the second part of (ii) is obtained by interchanging the roles of \(x'\) and \(x''. \)

As usual \(S\) denotes the class of all \(C^{\infty}\) functions of rapid decrease.

**Lemma 2.** Let \(\gamma \geq 0, \delta \geq 0, 1 < p < \infty\) be fixed. The equation

\[ \int_{\mathbb{R}^n} G_{\gamma + \delta}(x - y) g(y) \, dy \]

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{\gamma}'(x' - y') G_{\delta}'(x'' - y'') f(y', y'') \, dy' dy'' \]

(5)
establishes a 1-1 mapping \( f \rightarrow g \) of \( S \) onto itself. There exists a constant \( C \) such that

1. \( \| f \|_{\nu, p, k} \leq C \| g \|_{\nu, p, k} \) for all \( g \in S, \ 0 < \eta < k \).

**Proof.** Rewriting (5) in terms of Fourier transforms and using (1) we get

\[
\tilde{f}(\xi) = \frac{(1 + |\xi|^{2})^{\nu/2}(1 + |\xi|^2|\xi|^2)^{k/2}}{(1 + |\xi|^{2})^{(\nu+k)/2}} \tilde{g}(\xi).
\]

The first statement of the lemma is now trivial. Using the Mihlin theorem [3] about multipliers of Fourier transforms we verify immediately that the coefficient of \( \tilde{g}(\xi) \) on the right hand side of (6) is a multiplier of type \((p, p)\) for every \( p, k \), \( 1 < p < \infty \), which proves (i).

(ii) is obtained from (i) by replacing \( f \) and \( g \) by \( \Delta_{f}f \) and \( \Delta_{g}f \), respectively, with an arbitrary fixed \( t \in R^n \).

**Proof of the Theorem.** Let \( \gamma \geq 0, \ \delta \geq 0 \) and \( u, f, g \in S \) be related by the equation

\[
u(x', x'') = \int_{R^n} G_{\gamma+k}(x - y)g(y) \, dy
\]

\[
= \int_{R^n} \int_{R^n} G_{\gamma}^t (x' - y')G_{\delta}^t (x'' - y'')f(y', y'') \, dy' \, dy''
\]

\[
= \int_{R^n} G_{\gamma}^t (x' - y') \left[ \int_{R^n} G_{\delta}^t (x'' - y'')f(y', y'') \, dy'' \right] \, dy'.
\]

Let \( \alpha > 0 \) and \( 0 \leq \beta \leq \alpha \); we put, in (7), \( \delta = \beta, \ \gamma = \alpha - \beta \). By the definition of the norm \( \| \cdot \|_{\beta, p} \), we have, for every \( x' \in R^n \),

\[
\| u(x') \|_{\beta, p} = \left\| \int_{R^n} G_{\beta}^t(x' - y')f(y', y'') \, dy' \right\|_{L^p}
\]

and using the continuous version of the Minkowski inequality

\[
\| u(x') \|_{\beta, p} \leq \int_{R^n} G_{\beta}^t(x' - y') \| f(y', y'') \|_{L^p(R^n)} \, dy'
\]

\[
= \int_{R^n} G_{\beta}^t(x' - y')f(y') \, dy'.
\]

Let now \( \alpha > 0, \ 0 < \beta \leq \alpha \) and \( \epsilon = (1/2) \min(1, \beta) \). Choose, in (7), \( \delta = \beta - \epsilon, \ \gamma = \alpha - \beta \). Using Proposition 2, we get, with an integer \( k > \beta \) and a constant \( C \) depending only on \( \beta, \) and \( \eta' \),

\[
\| u(x') \|_{\beta, p, k} \leq C \left\| \int_{R^n} G_{\beta}^t(x' - y')f(y', y'') \, dy' \right\|_{L^p, 1},
\]
and, using again a continuous form of the Minkowski inequality,

\[ \|u_x\|_{\beta,p,k} \lesssim C \left[ \left( \int_{\mathbb{R}^n} G_{\alpha-\beta}(x'-y')f'(y') \, dy' \right)^p + \left( \int_{\mathbb{R}^n} G_{\alpha-\beta}(x'-y')f''(y') \, dy' \right)^p \right]^{1/p} \]

(9)

\[ \lesssim C \left[ \int_{\mathbb{R}^n} G_{\alpha-\beta}(x'-y') \left[ f'(y') + f''(y') \right] \, dy' \right]. \]

Finally, let \( \alpha > 0, \, 0 \leq \beta < \alpha \) and \( \epsilon = (1/2) \min(1, \alpha - \beta) \). Put, in (7), \( \delta = \beta, \, \gamma = \alpha - \beta - \epsilon \). We get, similarly, as in (8),

\[ \|u_x\|_{\beta,p} \lesssim \int_{\mathbb{R}^n} G_{\alpha-\beta-\epsilon}(x'-y')f'(y') \, dy'. \]

(10)

In the inequalities (8), (9), and (10), \( f' \) and \( f'' \) are defined as in Lemma 1.

We shall now prove (ii). Let \( A \in \mathfrak{B}_{\alpha,p} \). By Proposition 3, there is a sequence \( \{u_i\} \subset C_0^\infty, \, u_i \geq 0, \) Cauchy in \( B^{\alpha,p} \) and such that \( \lim_{i \to \infty} u_i(x) = +\infty \) for all \( x \in A \). Let \( \varepsilon = (1/2) \min(1, \beta) \) and \( \{f_i\}, \, \{g_i\} \subset S \) be such that

\[ u_i(x) = \int_{\mathbb{R}^n} G_{\alpha-\beta}(x - y)g_i(y) \, dy \]

(11)

\[ = \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y') \int_{\mathbb{R}^n} G_{\beta}(x'' - y'' f_i(y', y'') \, dy'dy''. \]

By Proposition 2, we may assume, without loss of generality, that \( \sum_{i=1}^\infty \|g_{i+1} - g_i\|_{p,1} < \infty \), consequently, by Lemmas 1 and 2, \( h' = \sum_{i=1}^\infty (f_{i+1} - f_i) + \sum_{i=1}^\infty (f_{i+1} - f_i)_\epsilon \in L^p(\mathbb{R}^n) \). We conclude, using (9), that the sequence \( \{u_i\} \) is Cauchy in \( B^{\beta,p}(\mathbb{R}^n') \) for every \( x' \) outside of the set \( A' = \{x' \in \mathbb{R}^n' : \int_{\mathbb{R}^n} G_{\alpha-\beta}(x' - y')h'(y') \, dy' = +\infty \} \in \mathfrak{Y}_{\alpha-\beta,p}' \).

This proves, using the functional space property, that \( A_x \in \mathfrak{B}_{\alpha,p}' \) for \( x' \in A' \) i.e., \( \mathfrak{Y}_{\alpha-\beta,p}' \).

Let now \( u \in B^{\beta,p} \) and choose an integer \( k > \alpha \). By the functional space property there is a sequence \( \{u_i\} \subset C_0^\infty \) such that \( \lim_{i \to \infty} \|u - u_i\|_{\alpha,p,k} = 0 \) and \( \lim_{i \to \infty} u_i(x) = u(x) \) exc. \( \mathfrak{B}_{\alpha,p} \). Let \( A \in \mathfrak{B}_{\alpha,p} \) be the union of the exceptional set of \( u \) (i.e., the set where \( u \) is not defined or infinite) and the set where \( \{u_i(x)\} \) does not converge to \( u(x) \). Let \( A' \in \mathfrak{B}_{\alpha-\beta,p}' \) be the set with the property that \( A_x \in \mathfrak{B}_{\alpha,p}' \) for \( x' \in A' \). Assume without loss of generality that \( \sum_{i=1}^\infty \|u_{i+1} - u_i\|_{\alpha,p,k} < \infty \), define \( \{f_i\}, \, \{g_i\} \subset S \) as in (11) and let
\[ h' = \sum_{l=1}^{\infty} (f_{i+1} - f_i) + \sum_{l=1}^{\infty} (f_{i+1} - f_i)' \in L^p(\mathbb{R}^n). \]

It follows from (9) that \((u_\lambda)_{x'}\) is Cauchy in \(B^{\beta,p}(\mathbb{R}^{n'})\) for every \(x'\) outside of the set \(B' = \{ x' \in \mathbb{R}^{n'} : \int G_{\alpha-\beta}(x' - y')h'(y') \, dy' = + \infty \}. \)

Consequently, for \(x' \in A' \cup B', \ (u_\lambda)_{x'}\) is Cauchy in \(B^{\beta,p}(\mathbb{R}^{n'})\) and converges to \(u_{x'}\) exc. \(\mathbb{B}''\). Since \(A' \cup B' \in \mathcal{H}_{\alpha-\beta,p}\) this proves that \(u_{x'} \in B^{\beta,p}(\mathbb{R}^{n'})\) exc. \(\mathcal{H}_{\alpha-\beta,p}\). The proofs of the remaining part of (ii) and of (i) follow the same idea and are even simpler. We use the statements (i) of Lemmas 1 and 2 and inequality (8) to prove (i) and the statements (ii) of Lemmas 1, 2 and inequality (10) to prove the remainder of (ii).

References